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This paper closely examines S. E. Dickson's definition of a torsion theory for the category ${}_R\mathcal{M}$ of left R-modules over a ring R. The author gives useful characterizations of torsion theories and presents a method of generating a torsion theory from an arbitrary class of left R-modules. It is proved that there is a one-to-one correspondence between hereditary torsion theories for ${}_R\mathcal{M}$ and torsion filters for the ring R. Further it is shown that a hereditary torsion class $\mathcal{T}$ is a TTF class if and only if its associated filter $F(\mathcal{T})$ has a smallest element. The paper concludes with the construction and investigation of three specific torsion theories: the E(R)-torsion theory, the Goldie torsion theory, and the simple torsion theory.

A GENERALIZATION OF TORSION TO MODULES

by

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INTRODUCTION AND PRELIMINARY REMARKS

The concepts of torsion and torsion-free objects have their origins in abelian group theory, where for an abelian group G the torsion subgroup $T(G)$ of G is defined by $T(G) = \{x \in G \mid$ there exists a positive integer n such that $nx = 0\}$, and where G is torsion-free provided $T(G) = 0$. Several ways of generalizing these notions to the category ${}_R\mathcal{M}$ of left R-modules over a ring are known. In [3] Dickson defines the concept of a torsion theory for certain abelian categories which include ${}_R\mathcal{M}$, and this definition encompasses most of the standard generalizations of torsion and torsion-free in ${}_R\mathcal{M}$. We shall study these torsion theories in ${}_R\mathcal{M}$ rather than in the more general setting in which they originally appeared.

In Chapter I we define a torsion theory $(\mathcal{T},\mathcal{F})$ for ${}_R\mathcal{M}$ and study the characterizations of torsion theories. The fundamental definitions and theorems which will be needed throughout the remainder of this work are stated and proved. Our main result is that a class $\mathcal{T}$ of modules is a torsion class if and only if $\mathcal{T}$ is closed under homomorphic images, extensions, and arbitrary direct sums. Dually we have that a class $\mathcal{F}$ of modules is a torsion-free class if and only if $\mathcal{F}$ is closed under submodules, extensions, and arbitrary direct products. We also show that in a torsion theory $(\mathcal{T},\mathcal{F})$ for ${}_R\mathcal{M}$, $\mathcal{T}$ and $\mathcal{F}$ uniquely determine each other, and we exhibit two ways of characterizing the torsion submodule of a module.

In Chapter II we look at ways of generating torsion theories. We prove $\mathcal{T} = \{M \mid \mathrm{Hom}(M,Y_0) = 0\}$, where Y_0 is an injective module , is a hereditary torsion class. We define operators L and R on a class of modules and exhibit a way to generate a torsion class $LR(\mathcal{A})$ from an arbitrary class $\mathcal{A}$ of left R-modules. We prove that $LR(\mathcal{A})$ is the smallest torsion class containing $\mathcal{A}$ and find conditions on $\mathcal{A}$ sufficient for $LR(\mathcal{A})$ to be hereditary. Finally we examine the behavior of these operators under certain types of unions and intersections.

In Chapter III we show that each hereditary torsion class for ${}_R\mathcal{M}$ gives rise to a torsion filter $F(\mathcal{T})$ of left ideals of the ring R, and dually each torsion filter for R yields a hereditary torsion class for ${}_R\mathcal{M}$. In fact, we prove there is a one-to-one correspondence between hereditary torsion theories for ${}_R\mathcal{M}$ and torsion filters for R.

In Chapter IV we take a look at a special type of hereditary torsion class, first introduced by Jans [4], called a TTF class. We prove that a hereditary torsion class is a TTF class if and only if its associated torsion filter has a smallest element. We further show that this smallest element can have a profound influence on the behavior of the torsion class; specifically, we obtain strong conditions which imply that the torsion submodule of any module is a direct summand of that module.

In Chapter V we construct three specific hereditary torsion theories for ${}_R\mathcal{M}$; namely, the E(R)-torsion theory, the Goldie torsion theory, and the simple torsion theory. We compute the filters for each of the torsion classes, we show when each torsion theory coincides with the standard concept of torsion for a module over an integral domain, and we investigate inclusion relationships among these torsion theories.

Throughout this thesis, the term "ring" will mean a ring with unit 1, and all modules are assumed to be unitary. We will deal exclusively with left R-modules and so will write R-module with the understanding that left is intended. When no confusion results we will omit specific reference to the ring R. Thus "R-module" becomes "module", "$\mathrm{Hom}_R(M,N)$" becomes "Hom(M,N)", and so on. Following current practice, ${}_R\mathcal{M}$ will denote the category of unitary left R-modules.

If M and N are modules, ${}_RN \leq {}_RM$ or just $N \leq M$ means that N is a left R-submodule of the left R-module M. Thus ${}_RI \leq {}_RR$ means that I is a left ideal of R. When set inclusion only is intended, we shall write $N \subseteq M$. If ${}_RN \leq {}_RM$ and if $x \in M$, then $(N:x) = \{r \in R \mid rx \in N\}$; $(N:x)$ is easily seen to be a left ideal of R.

If $N \leq M$, we say N is <u>essential</u> in M, and write $N \trianglelefteq M$, provided $N \cap K \neq 0$ for all $0 \neq K \leq M$. For any module M there is an injective module $E_R(M)$ which contains an essential submodule isomorphic to M. The module $E(M)$ is called the <u>injective envelope</u> or <u>injective hull</u> of M, and it is unique up to isomorphism. We will assume M is actually a submodule of $E(M)$.

For a module M, the <u>singular submodule</u> $Z_R(M)$ of M is defined by $Z_R(M) = \{x \in M \mid (0:x)$ is an essential left ideal of $R\}$.

One can check that $Z_R(M)$ is indeed a submodule of M. When no confusion results we will usually write $Z(M)$ for $Z_R(M)$. Now let $Z_1(M) = Z(M)$ and for $i > 1$ we let $Z_i(M)$ be defined by $Z_i(M)/Z_{i-1}(M) = Z(M/Z_{i-1}(M))$. From this we see that $Z_i(M) = \{x \in M \mid (Z_{i-1}(M):x) \trianglelefteq R\}$.

A module S is called simple provided its only submodules are 0 and S.

A left ideal I of R is said to be dense in R provided $(I:a) \cdot b = 0$ implies $b = 0$ for all $a,b \in R$.

By an integral domain R' we mean a commutative ring with unit 1 with the property that $ab = 0$ implies $a = 0$ or $b = 0$ for all $a,b \in R'$. The standard concept of torsion for a module M over an integral domain R' is to define the torsion submodule T(M) of M by $T(M) = \{x \in M \mid \text{there exists } 0 \neq r \in R' \text{ such that } rx = 0\}$.

We shall often refer to various "closure" properties of classes of modules. Let $\mathcal{A}$ be a class of modules; we say that

(a) $\mathcal{A}$ is closed under homomorphic images if $A \in \mathcal{A}$ and $A \to B \to 0$ exact implies that $B \in \mathcal{A}$;

(b) $\mathcal{A}$ is closed under submodules if $A \in \mathcal{A}$ and $0 \to B \to A$ exact implies that $B \in \mathcal{A}$;

(c) $\mathcal{A}$ is closed under extensions if $A,C \in \mathcal{A}$ and $0 \to A \to B \to C \to 0$ exact implies that $B \in \mathcal{A}$;

(d) $\mathcal{A}$ is closed under direct sums (direct products) if $A_i \in \mathcal{A}$ for each $i \in I$ implies that $\oplus\sum_I A_i \in \mathcal{A}$ ($\prod_I A_i \in \mathcal{A}$);

(e) $\mathcal{A}$ is <u>closed under injective envelopes</u> if $A \in \mathcal{A}$ implies that $E(A) \in \mathcal{A}$.

We will indicate the conclusion of a proof by the symbol $\square$.

CHAPTER I

CHARACTERIZATIONS OF TORSION THEORIES

1.1 DEFINITION. A torsion theory for ${}_R\mathcal{M}$ is a pair $(\mathcal{T},\mathcal{F})$ of classes of modules satisfying:

(a) $\mathcal{T} \cap \mathcal{F} = 0$;

(b) $\mathcal{T}$ is closed under homomorphic images and $\mathcal{F}$ is closed under submodules;

(c) For each module M there exists a submodule M_t of M such that $M_t \in \mathcal{T}$ and $M/M_t \in \mathcal{F}$.

The modules in $\mathcal{T}$ are called torsion modules, and the modules in $\mathcal{F}$ are called torsion-free modules. The submodule M_t of M is called the torsion submodule of M.

If $\mathcal{T}$ is a class of modules, then $\mathcal{T}$ is a torsion class provided there is a class $\mathcal{F}$ of modules such that $(\mathcal{T},\mathcal{F})$ is a torsion theory. If $\mathcal{F}$ is a class of modules, then $\mathcal{F}$ is a torsion-free class provided there is a class $\mathcal{T}$ of modules such that $(\mathcal{T},\mathcal{F})$ is a torsion theory. A torsion class $\mathcal{T}$, and the associated torsion theory $(\mathcal{T},\mathcal{F})$, is called hereditary if $M \in \mathcal{T}$ and $N \leq M$ imply that $N \in \mathcal{T}$.

1.2 THEOREM. Let $(\mathcal{T},\mathcal{F})$ be a torsion theory for ${}_R\mathcal{M}$. If $T \in \mathcal{T}$ and $M \cong T$, then $M \in \mathcal{T}$. Dually, if $F \in \mathcal{F}$ and $N \cong F$, then $N \in \mathcal{F}$.

Proof: Let $T \in \mathcal{T}$ and $M \cong T$. Then $T \to M \to 0$ is an exact sequence, and $M \in \mathcal{T}$ since $\mathcal{T}$ is closed under homomorphic images. The dual follows similarly. □

1.3 THEOREM. Let $(\mathcal{T},\mathcal{F})$ be a torsion theory for ${}_R\mathcal{M}$. Then $\mathcal{T}$ and $\mathcal{F}$ uniquely determine each other. Specifically,
$\mathcal{T} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,F) = 0 \text{ for all } F \in \mathcal{F}\}$ and
$\mathcal{F} = \{N \in {}_R\mathcal{M} \mid \mathrm{Hom}(T,N) = 0 \text{ for all } T \in \mathcal{T}\}$.

Proof: We will first verify $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,F) = 0$ for all $F \in \mathcal{F}\}$. Let $T \in \mathcal{T}$. Let $F \in \mathcal{F}$ and let $f \in \mathrm{Hom}(T,F)$. The sequence $T \to \mathrm{Im}\, f \to 0$ is exact, and $\mathrm{Im}\, f \in \mathcal{T}$ since $\mathcal{T}$ is closed under homomorphic images. The sequence $0 \to \mathrm{Im}\, f \to F$ is also exact, and $\mathrm{Im}\, f \in \mathcal{F}$ since $\mathcal{F}$ is closed under submodules. We now have $\mathrm{Im}\, f \in \mathcal{T} \cap \mathcal{F} = 0$ and $f = 0$. Therefore $\mathrm{Hom}(T,F) = 0$, and we have verified $\mathcal{T} \subseteq \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,F) = 0$ for all $F \in \mathcal{F}\}$.

Now let $\mathrm{Hom}(M,F) = 0$ for all $F \in \mathcal{F}$. Since $(\mathcal{T},\mathcal{F})$ is a torsion theory, for the module M there exists a submodule M_t of M such that $M_t \in \mathcal{T}$ and $M/M_t \in \mathcal{F}$. Consider the exact sequence $0 \to M_t \to M \xrightarrow{g} M/M_t \to 0$ where g is the natural homomorphism. Since $M/M_t \in \mathcal{F}$, $\mathrm{Hom}(M,M/M_t) = 0$ and $g = 0$. Thus $M/M_t = \mathrm{Im}\, g = 0$ and $M = M_t \in \mathcal{T}$. Therefore $\{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,F) = 0$ for all $F \in \mathcal{F}\} \subseteq \mathcal{T}$.

We have now shown $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,F) = 0$ for all $F \in \mathcal{F}\}$. Similarly, $\mathcal{F} = \{N \in {}_R\mathcal{M} \mid \mathrm{Hom}(T,N) = 0$ for all $T \in \mathcal{T}\}$. It then follows immediately that $\mathcal{T}$ and $\mathcal{F}$ uniquely determine each other. □

1.4 THEOREM. A class $\mathcal{T}$ of modules is a torsion class if and only if $\mathcal{T}$ is closed under homomorphic images, extensions, and arbitrary direct sums.

Proof: ($\rightarrow$) Let $\mathcal{T}$ be a class of modules which is a torsion class. Therefore, there exists a class $\mathcal{F}$ of modules such that $(\mathcal{T},\mathcal{F})$ is a torsion theory. We shall now show $\mathcal{T}$ is closed under (a) homomorphic images, (b) extensions, and (c) arbitrary direct sums.

(a) This follows immediately from the fact $(\mathcal{T},\mathcal{F})$ is a torsion theory.

(b) To show $\mathcal{T}$ is closed under extensions, let $0 \rightarrow T_1 \xrightarrow{f} A \xrightarrow{g} T_2 \rightarrow 0$ be an exact sequence with T_1 and T_2 in $\mathcal{T}$, and let $F \in \mathcal{F}$. Define $\phi : \mathrm{Hom}(T_2,F) \rightarrow \mathrm{Hom}(A,F)$ by if $\alpha \in \mathrm{Hom}(T_2,F)$, then $\phi(\alpha) = \alpha g$. Define $\psi : \mathrm{Hom}(A,F) \rightarrow \mathrm{Hom}(T_1,F)$ by if $\beta \in \mathrm{Hom}(A,F)$, then $\psi(\beta) = \beta f$. We now claim $0 \rightarrow \mathrm{Hom}(T_2,F) \xrightarrow{\phi} \mathrm{Hom}(A,F) \xrightarrow{\psi} \mathrm{Hom}(T_1,F)$ is exact. To verify this we must show (i) $\mathrm{Ker}\ \phi = 0$ and (ii) $\mathrm{Im}\ \phi = \mathrm{Ker}\ \psi$. Let $t_2 \in T_2$ and assume $\alpha \in \mathrm{Hom}(T_2,F)$ such that $\alpha g = 0$. Then since g is onto there exists an $a \in A$ such that $g(a) = t_2$. Thus $\alpha(t_2) = \alpha g(a) = 0$, so that $\alpha = 0$. We now have $\mathrm{Ker}\ \phi = \{\alpha \in \mathrm{Hom}(T_2,F) \mid \alpha g = 0\} = 0$. Thus (i) is verified. Now let $h \in \mathrm{Im}\ \phi$. Then $h = \alpha g$ for some $\alpha \in \mathrm{Hom}(T_2,F)$. But $\psi(h) = hf = \alpha gf = 0$, so $h \in \mathrm{Ker}\ \psi$ and we have $\mathrm{Im}\ \phi \subseteq \mathrm{Ker}\ \psi$. Now let $\beta \in \mathrm{Ker}\ \psi$. Then $\beta f = \psi(\beta) = 0$ and $\mathrm{Ker}\ g = \mathrm{Im}\ f \subseteq \mathrm{Ker}\ \beta$.

Define $\alpha : T_2 \to F$ by $\alpha(x) = \beta(\bar{x})$ where $g(\bar{x}) = x$. To verify α is well-defined, let $x_1, x_2 \in T_2$ such that $x_1 = x_2$. There exist $\bar{x}_1$ and $\bar{x}_2$ in A such that $g(\bar{x}_1) = x_1$ and $g(\bar{x}_2) = x_2$. Thus $g(\bar{x}_1) = g(\bar{x}_2) \Rightarrow g(\bar{x}_1 - \bar{x}_2) = 0 \Rightarrow \bar{x}_1 - \bar{x}_2 \in \operatorname{Ker} g \subseteq \operatorname{Ker} \beta \Rightarrow \beta(\bar{x}_1 - \bar{x}_2) = 0 \Rightarrow \beta(\bar{x}_1) = \beta(\bar{x}_2) \Rightarrow \alpha(x_1) = \alpha(x_2)$, and we have α is well-defined. Now if $\bar{x} \in A$ then $\alpha g(\bar{x}) = \beta(\bar{x})$, and thus $\beta \in \operatorname{Im} \phi$. Therefore $\operatorname{Ker} \psi \subseteq \operatorname{Im} \phi$. We now have $\operatorname{Im} \phi = \operatorname{Ker} \psi$, and this completes (ii). Thus $0 \to \operatorname{Hom}(T_2,F) \xrightarrow{\phi} \operatorname{Hom}(A,F) \xrightarrow{\psi} \operatorname{Hom}(T_1,F)$ is exact. Now $\operatorname{Hom}(T_2,F)$ and $\operatorname{Hom}(T_1,F)$ are both zero by 1.3, and so $\operatorname{Hom}(A,F) = 0$. Applying 1.3 again, we have $A \in \mathcal{T}$.

(c) To show $\mathcal{T}$ is closed under arbitrary direct sums, let $\{T_i \mid i \in I\}$ be a collection of modules in $\mathcal{T}$. Let $F \in \mathcal{F}$. We now claim $\operatorname{Hom}(\oplus \sum_I T_i, F) \cong \prod_I \operatorname{Hom}(T_i,F)$. Define $\phi : \operatorname{Hom}(\oplus \sum_I T_i, F) \to \prod_I \operatorname{Hom}(T_i,F)$ as follows: if

$f \in \operatorname{Hom}(\oplus \sum_I T_i, F)$, then $\pi_i\ \phi(f) = f\theta_i$ where π_i and θ_i are the canonical projection and injection maps. Define $\psi : \prod_I \operatorname{Hom}(T_i,F) \to \operatorname{Hom}(\oplus \sum_I T_i, F)$ as follows: if $g \in \prod_I \operatorname{Hom}(T_i,F)$, then $\psi(g) = \delta$ where $\delta \in \operatorname{Hom}(\oplus \sum_I T_i, F)$ and $\delta(t) = \sum_I([\pi_i g](\pi_i(t)))$ for all $t \in \oplus \sum_I T_i$. One can verify that ϕ and ψ are well-defined homomorphisms. Now let $f \in \operatorname{Hom}(\oplus \sum_I T_i, F)$ and let $t \in \oplus \sum_I T_i$. Then $[\psi\ \phi(f)](t) = \sum_I([\pi_i\ \phi(f)](\pi_i(t))) = \sum_I([f\theta_i](\pi_i(t))) = \sum_I f\theta_i \pi_i(t) = f(t)$ and hence $\psi\ \phi(f) = f$. Now let $g \in \prod_I \operatorname{Hom}(T_i,F)$. Let $j \in I$ and $t_j \in T_j$. Then $[\pi_j\ \phi\ \psi(g)](t_j) = [\psi(g)\theta_j](t_j) = [\psi(g)](\theta_j(t_j)) =$

$\sum_I([\pi_i g](\pi_i \theta_j(t_j))) = [\pi_j g](t_j)$. Thus $\pi_j \phi \psi(g) = \pi_j g$ for every $j \in I$, and hence $\phi \psi(g) = g$. Therefore $\mathrm{Hom}(\bigoplus \sum_I T_i, F) \cong \prod_I \mathrm{Hom}(T_i, F)$, and by 1.3 $\mathrm{Hom}(T_i, F) = 0$ for all $i \in I$ so $\prod_I \mathrm{Hom}(T_i, F) \cong 0$. Thus $\mathrm{Hom}(\bigoplus \sum_I T_i, F) = 0$, and applying 1.3 again we have $\bigoplus \sum_I T_i \in \mathcal{T}$.

($\leftarrow$) Let $\mathcal{T}$ be a class of modules which is closed under homomorphic images, arbitrary direct sums, and extensions. Let $\mathcal{F} = \{F \in {}_R\mathcal{M} \mid \mathrm{Hom}(T,F) = 0$ for all $T \in \mathcal{T}\}$. We want to show $(\mathcal{T}, \mathcal{F})$ is a torsion theory. Clearly $\mathcal{T} \cap \mathcal{F} = 0$ from the way we defined $\mathcal{F}$, and $\mathcal{T}$ is closed under homomorphic images by the hypothesis. Let M be a module and define $M_t = \Sigma\{T \leq M \mid T \in \mathcal{T}\}$. Clearly M_t is a submodule of M, and $M_t \in \mathcal{T}$ since $\mathcal{T}$ is closed under direct sums and homomorphic images. To show $M/M_t \in \mathcal{F}$, let $T \in \mathcal{T}$ and $f \in \mathrm{Hom}(T, M/M_t)$. The image of f is of the form H/M_t where H is a submodule of M containing M_t. The sequence $T \to H/M_t \to 0$ is exact, and $H/M_t \in \mathcal{T}$ since $\mathcal{T}$ is closed under homomorphic images. Now $0 \to M_t \to H \to H/M_t \to 0$ is exact, and since $\mathcal{T}$ is closed under extensions, $H \in \mathcal{T}$. Since $H \in \mathcal{T}$ and H is a submodule of M, then $H \subseteq M_t$, and thus $H = M_t$. We now have $\mathrm{Im}\, f = 0$ which implies $f = 0$, and thus $M/M_t \in \mathcal{F}$ by the way we defined $\mathcal{F}$. To complete the proof we must now verify $\mathcal{F}$ is closed under submodules. Let $0 \to A \to F$ be an exact sequence with $F \in \mathcal{F}$. We have just verified above that there exists a submodule A_t of A such that $A_t \in \mathcal{T}$ and $A/A_t \in \mathcal{F}$. The sequence $0 \to A_t \to F$ is exact, and by the way we defined $\mathcal{F}$, $\mathrm{Hom}(A_t, F) = 0$. Thus $A_t = 0$

and $A = A/A_t \varepsilon F$, so $A \varepsilon F$. Therefore by 1.1 (T,F) is a torsion theory. □

<u>1.5 THEOREM</u>. A class F of modules is a torsion-free class if and only if F is closed under submodules, arbitrary direct products, and extensions.

<u>Proof</u>: ($\rightarrow$) Let F be a class of modules which is a torsion-free class. Therefore, there exists a class T of modules such that (T,F) is a torsion theory. We already have F is closed under submodules since (T,F) is a torsion theory. To show F is closed under extensions, let $0 \rightarrow F_1 \rightarrow B \rightarrow F_2 \rightarrow 0$ be an exact sequence with F_1 and F_2 in F. Let $T \varepsilon T$. Following the pattern of the proof in 1.4, one can show that the sequence $\mathrm{Hom}(T,F_1) \rightarrow \mathrm{Hom}(T,B) \rightarrow \mathrm{Hom}(T,F_2) \rightarrow 0$ is exact. Now $\mathrm{Hom}(T,F_1)$ and $\mathrm{Hom}(T,F_2)$ are both zero by 1.3, so $\mathrm{Hom}(T,B) = 0$. By applying 1.3 again we have $B \varepsilon F$. To show F is closed under arbitrary direct products, let $\{F_i \mid i \varepsilon I\}$ be a collection of modules in F. Let $T \varepsilon T$. Again following the pattern in the proof of 1.4, one can show $\mathrm{Hom}(T,\prod_I F_i) \cong \prod_I \mathrm{Hom}(T,F_i) \cong 0$, and thus by 1.3 we have $\prod_I F_i \varepsilon F$.

($\leftarrow$) Let F be a class of modules which is closed under submodules, arbitrary direct products, and extensions. Let $T = \{T \varepsilon {}_R M \mid \mathrm{Hom}(T,F) = 0$ for all $F \varepsilon F\}$. We want to show (T,F) is a torsion theory. Clearly $T \cap F = 0$ from the way we defined T, and F is closed under submodules by hypothesis. Let

M be a module and define $M_t = \cap \{K \leq M \mid M/K \in \mathcal{F}\}$. Clearly M_t is a submodule of M, and we now need to verify (i) $M/M_t \in \mathcal{F}$ and (ii) $M_t \in \mathcal{T}$. Let $\{K_i \mid i \in I\}$ be the collection of submodules of M such that $M/K_i \in \mathcal{F}$. Define $f : M/M_t \to \prod_I M/K_i$ by if $m + M_t \in M/M_t$, then $\pi_i f(m+M_t) = m + K_i$. One can verify that f is well-defined and one-to-one, so $0 \to M/M_t \xrightarrow{f} \prod_I M/K_i$ is an exact sequence. By hypothesis $\prod_I M/K_i \in \mathcal{F}$ and thus $M/M_t \in \mathcal{F}$. Thus (i) has been verified. Now let $F \in \mathcal{F}$ and $g \in \mathrm{Hom}(M_t,F)$. Then $\mathrm{Im}\, g \subseteq F$, and we have $\mathrm{Im}\, g \in \mathcal{F}$. Also by the first isomorphism theorem $M_t/\mathrm{Ker}\, g \cong \mathrm{Im}\, g$, and $M_t/\mathrm{Ker}\, g \in \mathcal{F}$ by 1.2. Now consider the exact sequence $0 \to M_t/\mathrm{Ker}\, g \to M/\mathrm{Ker}\, g \to M/M_t \to 0$. Since $\mathcal{F}$ is closed under extensions, we now know $M/\mathrm{Ker}\, g \in \mathcal{F}$, and this implies $M_t \subseteq \mathrm{Ker}\, g$. Thus $g = 0$ and $M_t \in \mathcal{T}$ by the way we defined $\mathcal{T}$, and hence we have completed (ii). To finish the proof we now must verify $\mathcal{T}$ is closed under homomorphic images. Let $T \to A \to 0$ be an exact sequence with $T \in \mathcal{T}$. We have just shown that there exists a submodule A_t of A such that $A_t \in \mathcal{T}$ and $A/A_t \in \mathcal{F}$. The sequence $T \to A/A_t \to 0$ is exact, and by the way we defined $\mathcal{T}$, $\mathrm{Hom}(T,A/A_t) = 0$. Thus $A/A_t = 0$ and $A = A_t \in \mathcal{T}$, so $A \in \mathcal{T}$. Therefore by 1.1, $(\mathcal{T},\mathcal{F})$ is a torsion theory. □

<u>1.6 COROLLARY</u>. Let $(\mathcal{T},\mathcal{F})$ be a torsion theory for ${}_R\mathcal{M}$. For each $M \in {}_R\mathcal{M}$ we have $M_t = \Sigma \{T \leq M \mid T \in \mathcal{T}\}$ and $M_t = \cap \{K \leq M \mid M/K \in \mathcal{F}\}$.

Proof: This follows directly from the proofs of 1.4 and 1.5. □

1.7 REMARK. Let $(\mathcal{T},\mathcal{F})$ be a torsion theory for $_R\mathcal{M}$ and let M be a module. With the above characterization that $M_t = \Sigma\{T \leq M \mid T \in \mathcal{T}\}$, it becomes clear that M_t is the least upper bound of $\{T \leq M \mid T \in \mathcal{T}\}$ and that M_t is necessarily unique.

1.8 THEOREM. Let $(\mathcal{T},\mathcal{F})$ be a torsion theory for $_R\mathcal{M}$. Then $\mathcal{T}$ is hereditary if and only if $\mathcal{F}$ is closed under injective envelopes.

Proof: $(\rightarrow)$ Let $\mathcal{T}$ be a hereditary torsion class. Let $F \in \mathcal{F}$, and let E be the injective envelope of F. There exists a submodule E_t of E such that $E_t \in \mathcal{T}$ and $E/E_t \in \mathcal{F}$. We have that $E_t \cap F \leq F$, and since $\mathcal{F}$ is closed under submodules, $E_t \cap F \in \mathcal{F}$. Also $E_t \cap F \leq E_t$, and since $\mathcal{T}$ is hereditary, $E_t \cap F \in \mathcal{T}$. Thus $E_t \cap F \in \mathcal{T} \cap \mathcal{F} = 0$ which implies $E_t = 0$ since $F \trianglelefteq E$. Therefore $E = E/E_t \in \mathcal{F}$, and $\mathcal{F}$ is closed under injective envelopes.

$(\leftarrow)$ Let $\mathcal{F}$ be closed under injective envelopes. Let $T \in \mathcal{T}$ and $A \leq T$. We wish to show $A \in \mathcal{T}$. Let $F \in \mathcal{F}$ and $E(F)$ the injective envelope of F. Let $f \in \mathrm{Hom}(A,F)$. Consider the following diagram with exact row:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_A"] \arrow[d, "f"'] & T \arrow[ddl, dashed, "\bar{f}"] \\
& F \arrow[d, "i_F"'] & \\
& E(F) &
\end{tikzcd}$$

Since $E(F)$ is injective, there exists an $\bar{f} \in \mathrm{Hom}(T,E(F))$ such that the diagram commutes; that is, $\bar{f}i_A = i_F f$. Since $\bar{f} = 0$, then $f = 0$, and $A \in \mathcal{T}$ by 1.3. Thus $\mathcal{T}$ is hereditary. □

CHAPTER II

GENERATION OF TORSION THEORIES

The following theorem gives us a method of generating a torsion theory for ${}_R\mathcal{M}$.

<u>2.1 THEOREM</u>. Let Y be an injective module. Define $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,Y) = 0\}$. Then $\mathcal{T}$ is a hereditary torsion class.

<u>Proof</u>: To show $\mathcal{T}$ is a hereditary torsion class we must prove $\mathcal{T}$ is closed under (i) submodules, (ii) homomorphic images, (iii) extensions, and (iv) arbitrary direct sums.

(i) Let $0 \to A \xrightarrow{f} B$ be an exact sequence with $B \in \mathcal{T}$. Let $g \in \mathrm{Hom}(A,Y)$. Since Y is injective there exists an $h \in \mathrm{Hom}(B,Y)$ such that $g = hf$. Let $a \in A$. Then since $h = 0$, $g(a) = hf(a) = h(f(a)) = 0$ and hence $g = 0$. Thus $\mathrm{Hom}(A,Y) = 0$ and $A \in \mathcal{T}$. Therefore $\mathcal{T}$ is closed under submodules.

(ii) Let $A \xrightarrow{f} B \to 0$ be an exact sequence with $A \in \mathcal{T}$. Let $g \in \mathrm{Hom}(B,Y)$. Then $gf \in \mathrm{Hom}(A,Y)$ and hence $gf = 0$. Let $b \in B$. Since f is onto there exists an $a \in A$ such that $f(a) = b$. Now $g(b) = g(f(a)) = gf(a) = 0$. Thus $g = 0$, $\mathrm{Hom}(B,Y) = 0$, $B \in \mathcal{T}$, and $\mathcal{T}$ is closed under homomorphic images.

(iii) Let $0 \to A \to B \to C \to 0$ be an exact sequence with A and C in $\mathcal{T}$. As show in the proof of 1.4, the sequence $0 \to \mathrm{Hom}(C,Y) \to \mathrm{Hom}(B,Y) \to \mathrm{Hom}(A,Y)$ is exact. But $\mathrm{Hom}(C,Y)$ and $\mathrm{Hom}(A,Y)$ are both zero, so $\mathrm{Hom}(B,Y) = 0$, and thus $B \in \mathcal{T}$ and $\mathcal{T}$ is closed under extensions.

(iv) Let $\{T_i \mid i \in I\}$ be a collection of elements of $\mathcal{T}$. As shown in the proof of 1.4, $\mathrm{Hom}(\bigoplus\sum_I T_i, Y) \simeq \prod_I \mathrm{Hom}(T_i,Y) \cong 0$. Thus $\bigoplus\sum_I T_i \in \mathcal{T}$, and $\mathcal{T}$ is closed under arbitrary direct sums. □

We now define operators L and R, and the two theorems which follow will then enable us to exhibit a way to generate a torsion class $LR(\mathcal{A})$ from an arbitrary class $\mathcal{A}$ of left R-modules.

<u>2.2 DEFINITION</u>. Let $\mathcal{A}$ denote a class of left R-modules. We define operators L and R as follows:

$L(\mathcal{A}) = \{B \in {}_R\mathcal{M} \mid \mathrm{Hom}(B,A) = 0 \text{ for all } A \in \mathcal{A}\}$;

$R(\mathcal{A}) = \{B \in {}_R\mathcal{M} \mid \mathrm{Hom}(A,B) = 0 \text{ for all } A \in \mathcal{A}\}$.

<u>2.3 THEOREM</u>. Let $\mathcal{A}$ and $\mathcal{B}$ be classes of left R-modules.

(i) $\mathcal{A} \cap L(\mathcal{A}) = \{0\}$ and $\mathcal{A} \cap R(\mathcal{A}) = \{0\}$;

(ii) $\mathcal{A} \subseteq LR(\mathcal{A})$ and $\mathcal{A} \subseteq RL(\mathcal{A})$;

(iii) If $\mathcal{A} \subseteq \mathcal{B}$, then $L(\mathcal{B}) \subseteq L(\mathcal{A})$ and $R(\mathcal{B}) \subseteq R(\mathcal{A})$;

(iv) $LRL = L$ and $RLR = R$;

(v) Let $T = LR$ and $F = RL$. Then $T^2 = T$ and $F^2 = F$.

Proof: (i) Let $M \in \mathcal{A} \cap L(\mathcal{A})$. Then $M \in \mathcal{A}$, and $\mathrm{Hom}(M,M) = 0$, which implies $M = 0$. Thus $\mathcal{A} \cap L(\mathcal{A}) = \{0\}$. Likewise $\mathcal{A} \cap R(\mathcal{A}) = \{0\}$.

(ii) Let $M \in \mathcal{A}$. Let $B \in R(\mathcal{A})$ and then $\mathrm{Hom}(A,B) = 0$ for all $A \in \mathcal{A}$. Therefore $\mathrm{Hom}(M,B) = 0$, and since B was arbitrarily chosen, $\mathrm{Hom}(M,B) = 0$ for all $B \in R(\mathcal{A})$. Hence $M \in LR(\mathcal{A})$ and $\mathcal{A} \subseteq LR(\mathcal{A})$. Similarly $\mathcal{A} \subseteq RL(\mathcal{A})$.

(iii) Let $\mathcal{A} \subseteq \mathcal{B}$ and $M \in L(\mathcal{B})$. Then $\mathrm{Hom}(M,B) = 0$ for all $B \in \mathcal{B}$, and since $\mathcal{A} \subseteq \mathcal{B}$, $\mathrm{Hom}(M,A) = 0$ for all $A \in \mathcal{A}$. Thus $M \in L(\mathcal{A})$, and $L(\mathcal{B}) \subseteq L(\mathcal{A})$. Similarly $R(\mathcal{B}) \subseteq R(\mathcal{A})$.

(iv) By (ii) $\mathcal{A} \subseteq RL(\mathcal{A})$, and by now applying (iii), $LRL(\mathcal{A}) \subseteq L(\mathcal{A})$. Now let $M \in L(\mathcal{A})$ and $C \in RL(\mathcal{A})$. Then $\mathrm{Hom}(B,C) = 0$ for all $B \in L(\mathcal{A})$, so $\mathrm{Hom}(M,C) = 0$. But since C was arbitrarily chosen, $\mathrm{Hom}(M,C) = 0$ for all $C \in RL(\mathcal{A})$, and hence $M \in LRL(\mathcal{A})$. Thus $L(\mathcal{A}) \subseteq LRL(\mathcal{A})$, and now $LRL(\mathcal{A}) = L(\mathcal{A})$. Since $\mathcal{A}$ is an arbitrary class of left R-modules, $LRL = L$. Similarly $RLR = R$.

(v) By (iv) $RLR = R$, and by now applying (iii) $LRLR = LR$ which says $T^2 = T$. Similarly $F^2 = F$. □

2.4 DEFINITION. We call a class $\mathcal{A}$ of left R-modules T-closed if $T(\mathcal{A}) = \mathcal{A}$. Likewise, we call a class $\mathcal{A}$ of left R-modules F-closed if $F(\mathcal{A}) = \mathcal{A}$.

From this definition one can quickly observe that any image of L is T-closed and any image of R is F-closed.

The following theorem reveals that the concepts of torsion class and T-closed class are equivalent.

2.5 THEOREM. The following are equivalent for the pair $(\mathcal{T},\mathcal{F})$ of classes of left R-modules:

(i) $(\mathcal{T},\mathcal{F})$ is a torsion theory for $_R\mathcal{M}$;

(ii) $\mathcal{T}$ is T-closed with $R(\mathcal{T}) = \mathcal{F}$;

(iii) $\mathcal{F}$ is F-closed with $L(\mathcal{F}) = \mathcal{T}$;

(iv) $R(\mathcal{T}) = \mathcal{F}$ and $L(\mathcal{F}) = \mathcal{T}$.

Proof: (i → ii) Let $(\mathcal{T},\mathcal{F})$ be a torsion theory for $_R\mathcal{M}$. Then $M \in R(\mathcal{T}) \Leftrightarrow \mathrm{Hom}(A,M) = 0$ for all $A \in \mathcal{T} \Leftrightarrow M \in \mathcal{F}$ by 1.3, and thus $R(\mathcal{T}) = \mathcal{F}$. But now $M \in T(\mathcal{T}) \Leftrightarrow M \in LR(\mathcal{T}) \Leftrightarrow \mathrm{Hom}(M,B) = 0$ for all $B \in R(\mathcal{T}) \Leftrightarrow \mathrm{Hom}(M,B) = 0$ for all $B \in \mathcal{F} \Leftrightarrow M \in \mathcal{T}$. Thus $T(\mathcal{T}) = \mathcal{T}$, and $\mathcal{T}$ is T-closed with $R(\mathcal{T}) = \mathcal{F}$.

(ii → iii) Let $\mathcal{T}$ be a T-closed class with $R(\mathcal{T}) = \mathcal{F}$. Then $R(\mathcal{T}) = \mathcal{F} \Rightarrow LR(\mathcal{T}) = L(\mathcal{F}) \Rightarrow T(\mathcal{T}) = L(\mathcal{F}) \Rightarrow \mathcal{T} = L(\mathcal{F})$. Also $F(\mathcal{F}) = RL(\mathcal{F}) = R(\mathcal{T}) = \mathcal{F}$. Hence $\mathcal{F}$ is F-closed with $L(\mathcal{F}) = \mathrm{T}$.

(iii → iv) Let $\mathcal{F}$ be F-closed with $L(\mathcal{F}) = \mathcal{T}$. Then $L(\mathcal{F}) = \mathcal{T} \Rightarrow RL(\mathcal{F}) = R(\mathcal{T}) \Rightarrow F(\mathcal{F}) = R(\mathcal{T}) \Rightarrow \mathcal{F} = R(\mathcal{T})$. Therefore $R(\mathcal{T}) = \mathcal{F}$ and $L(\mathcal{F}) = \mathcal{T}$.

(iv → i) Let $R(\mathcal{T}) = \mathcal{F}$ and $L(\mathcal{F}) = \mathcal{T}$. Observe that if $\mathrm{Hom}(K,N) = 0$ for all $N \in \mathcal{F}$, then $K \in L(\mathcal{F}) = \mathcal{T}$. Also observe if $\mathrm{Hom}(M,K) = 0$ for all $M \in \mathcal{T}$, then $K \in R(\mathcal{T}) = \mathcal{F}$. To show $(\mathcal{T},\mathcal{F})$ is a torsion theory we shall first show $\mathcal{T}$ is closed under (a) homomorphic images, (b) arbitrary direct sums, and (c) extensions.

(a) Let $A \xrightarrow{f} B \to 0$ be an exact sequence with $A \in \mathcal{T}$. Let $N \in \mathcal{F}$ and let $g \in \mathrm{Hom}(B,N)$. Let $b \in B$. Then there exists an $a \in A$ such that $f(a) = b$. Now $g(b) = g(f(a)) = gf(a) = 0$, because $gf \in \mathrm{Hom}(A,N) = 0$ since $A \in \mathcal{T}$ and $N \in \mathcal{F}$. Therefore $g = 0$, $\mathrm{Hom}(B,N) = 0$, $B \in \mathcal{T}$, and $\mathcal{T}$ is closed under homomorphic images.

(b) Let $\{M_i \mid i \in I\}$ be a collection of elements of $\mathcal{T}$. Let $K \in \mathcal{F}$. Then as shown in proof of 1.4, $\mathrm{Hom}(\oplus \sum_I M_i, K) \cong \prod_I \mathrm{Hom}(M_i,K) \cong 0$, and thus $\oplus \sum_I M_i \in \mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary direct sums.

(c) Let $0 \to A \to B \to C \to 0$ be an exact sequence with A and C in $\mathcal{T}$. Let $N \in \mathcal{F}$. As in the proof of 1.4, $0 \to \mathrm{Hom}(C,N) \to \mathrm{Hom}(B,N) \to \mathrm{Hom}(A,N)$ is an exact sequence, and $\mathrm{Hom}(C,N)$ and $\mathrm{Hom}(A,N)$ are both zero. Thus $\mathrm{Hom}(B,N) = 0$ and $B \in \mathcal{T}$. Therefore $\mathcal{T}$ is closed under extensions.

We now have that $\mathcal{T}$ is a torsion class, and by 1.3 the torsion-free class corresponding to $\mathcal{T}$ is $\{N \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,N) = 0$ for all $M \in \mathcal{T}\} = \mathcal{R}(\mathcal{T}) = \mathcal{F}$. Thus $(\mathcal{T},\mathcal{F})$ is a torsion theory. □

<u>2.6 IMPORTANT REMARK</u>. Using 2.5 we now know how to generate a torsion theory from an arbitrary class of left R-modules. Let $\mathcal{A}$ be a class of left R-modules and form $\mathcal{L}(\mathcal{A})$ and $\mathcal{R}(\mathcal{A})$. Let $\mathcal{L}(\mathcal{A}) = \mathcal{T}$ and $\mathcal{RL}(\mathcal{A}) = \mathcal{F}$. Then $T(\mathcal{T}) = \mathcal{LR}(\mathcal{T}) = \mathcal{LRL}(\mathcal{A}) = \mathcal{L}(\mathcal{A}) = \mathcal{T}$, so $\mathcal{T}$ is T-closed, and also $\mathcal{R}(\mathcal{T}) = \mathcal{RL}(\mathcal{A}) = \mathcal{F}$. Thus by 2.5, $(\mathcal{T},\mathcal{F})$ is a torsion theory for ${}_R\mathcal{M}$ with $\mathcal{A} \subseteq \mathcal{F}$. After we formed

$L(\mathcal{A})$ and $R(\mathcal{A})$, we could have let $R(\mathcal{A}) = \mathcal{F}'$ and $LR(\mathcal{A}) = \mathcal{T}'$ and thus obtained another torsion theory $(\mathcal{T}',\mathcal{F}')$ for ${}_R\mathcal{M}$ in the same manner with $\mathcal{A} \subseteq \mathcal{T}'$. One should observe that $(\mathcal{T}',\mathcal{F}')$ will be different from $(\mathcal{T},\mathcal{F})$ unless $LR(\mathcal{A}) = L(\mathcal{A})$ and $R(\mathcal{A}) = RL(\mathcal{A})$.

We can also verify that $LR(\mathcal{A})$ is the smallest torsion class containing $\mathcal{A}$. If $\mathcal{T}$ is any other torsion class containing $\mathcal{A}$, then by 2.3 (iii) and 2.5 $R(\mathcal{T}) \subseteq R(\mathcal{A})$ and $LR(\mathcal{A}) \subseteq LR(\mathcal{T}) = T(\mathcal{T}) = \mathcal{T}$. Similarly one can verify that $RL(\mathcal{A})$ is the smallest torsion-free class containing $\mathcal{A}$.

Now let $\mathcal{C}$ be a class of left R-modules closed under homomorphic images and let $\mathcal{T}_{\mathcal{C}} = \{M \in {}_R\mathcal{M} \mid$ every nonzero homomorphic image of M contains a nonzero submodule in $\mathcal{C}\}$. The following theorem verifies that $\mathcal{T}_{\mathcal{C}}$ is another characterization of the smallest torsion class containing $\mathcal{C}$.

<u>2.7 THEOREM</u>. $\mathcal{T}_{\mathcal{C}} = LR(\mathcal{C})$.

<u>Proof</u>: Let $M \in \mathcal{T}_{\mathcal{C}}$ and let $B \in R(\mathcal{C})$ and $f \in \mathrm{Hom}(M,B)$. Assume $f \neq 0$ and thus $\mathrm{Im}\, f \neq 0$. Thus there exists $0 \neq A \subseteq \mathrm{Im}\, f$ with $A \in \mathcal{C}$. So $\mathrm{Hom}(A,B) \neq 0$ but this contradicts the fact $B \in R(\mathcal{C})$. Thus $f = 0$, $M \in LR(\mathcal{C})$, and $\mathcal{T}_{\mathcal{C}} \subseteq LR(\mathcal{C})$.

Now let $M \in LR(\mathcal{C})$. Let D be a nonzero homomorphic image of M. Then $D \cong M/M'$ where $M' \leq M$. To show $M \in \mathcal{T}_{\mathcal{C}}$ we must show M/M' contains a nonzero submodule in $\mathcal{C}$. Since $M \in LR(\mathcal{C})$, $\mathrm{Hom}(M,B) = 0$ for all $B \in R(\mathcal{C})$ and since M/M' is a nonzero homomorphic image of M, $M/M' \notin R(\mathcal{C})$. Thus there exists an $A \in \mathcal{C}$

and an $f \in \mathrm{Hom}(A, M/M')$ such that $f \neq 0$. Hence $\mathrm{Im}\, f$ is a nonzero submodule of M/M' and $\mathrm{Im}\, f \cong A/\mathrm{Ker}\, f \in \mathcal{C}$ since $\mathcal{C}$ is closed under homomorphic images. Thus $M \in \mathcal{T}_{\mathcal{C}}$ and $LR(\mathcal{C}) \subseteq \mathcal{T}_{\mathcal{C}}$. Therefore $\mathcal{T}_{\mathcal{C}} = LR(\mathcal{C})$. $\square$

<u>2.8 THEOREM</u>. Let $\mathcal{C}$ be closed under homomorphic images and cyclic submodules. Then $\mathcal{T}_{\mathcal{C}}$ is a hereditary torsion class.

<u>Proof</u>: By the preceding theorem we already know that $\mathcal{T}_{\mathcal{C}}$ is a torsion class. Now let $B \in \mathcal{T}_{\mathcal{C}}$ and $A \leq B$. Let $f : A \to M$ be a nonzero epimorphism and $E(M)$ the injective hull of M. Then there is a map $f' : B \to E(M)$ which extends f. Since $B \in \mathcal{T}_{\mathcal{C}}$, $f'(B)$ has a nonzero submodule $C \in \mathcal{C}$. Observe $C \cap M \neq 0$ since $M \trianglelefteq E(M)$ and let $0 \neq x \in C \cap M$. Then Rx is a nonzero cyclic submodule of $C \cap M$ and $Rx \in \mathcal{C}$ since $\mathcal{C}$ is closed under cyclic submodules. Thus M has a nonzero submodule in $\mathcal{C}$, and this implies $A \in \mathcal{T}_{\mathcal{C}}$. $\square$

From 2.7 and 2.8 we can observe that $LR(\mathcal{A})$, the smallest torsion class containing a class $\mathcal{A}$ of modules, is a hereditary torsion class when $\mathcal{A}$ is closed under cyclic submodules and homomorphic images.

An interesting question about torsion theories is that if we have a collection $(\mathcal{T}_u, \mathcal{F}_u)$, $u \in U$, of torsion theories for ${}_R M$, what new torsion theories can we form using $\cup \mathcal{T}_u$, $\cap \mathcal{T}_u$, $\cup \mathcal{F}_u$, and $\cap \mathcal{F}_u$. With the help of our L and R operators, together with 2.5, we can find the answer to this question without much difficulty. The following lemmas and theorem will give us what we want.

<u>2.9 LEMMA</u>. Let $\{A_u \mid u \in U\}$ be a collection of classes of R-modules. Then $L(\cup A_u) = L(\cup RL(A_u)) = \cap L(A_u)$. Dually, $R(\cup A_u) = R(\cup LR(A_u)) = \cap R(A_u)$.

<u>Proof</u>: Let $u \in U$. Then $A_u \subseteq \cup A_u \Rightarrow L(\cup A_u) \subseteq L(A_u) \Rightarrow RL(A_u) \subseteq RL(\cup A_u) \Rightarrow \cup RL(A_u) \subseteq RL(\cup A_u) \Rightarrow L(\cup A_u) = LRL(\cup A_u) \subseteq L(\cup RL(A_u))$. Also $A_u \subseteq RL(A_u) \Rightarrow \cup A_u \subseteq \cup RL(A_u) \Rightarrow L(\cup RL(A_u)) \subseteq L(\cup A_u)$. Therefore $L(\cup A_u) = L(\cup RL(A_u))$.

Again let $u \in U$. Then $A_u \subseteq RL(A_u) \subseteq \cup RL(A_u) \Rightarrow L(\cup RL(A_u)) \subseteq L(A_u) \Rightarrow L(\cup RL(A_u)) \subseteq \cap L(A_u)$. Now let $M \in \cap L(A_u)$. Then $M \in L(A_u)$ for each $u \in U$. Let $C \in \cup RL(A_u)$. Then $C \in RL(A_{u'})$ for some $u' \in U$. Hence $\mathrm{Hom}(A,C) = 0$ for all $A \in L(A_{u'})$. Therefore $\mathrm{Hom}(M,C) = 0$, and since C was arbitrarily chosen, $\mathrm{Hom}(M,C) = 0$ for all $C \in \cup RL(A_u)$. Hence $M \in L(\cup RL(A_u))$ and $\cap L(A_u) \subseteq L(\cup RL(A_u))$. Therefore $L(\cup RL(A_u)) = \cap L(A_u)$. We now have $L(\cup A_u) = L(\cup RL(A_u)) = \cap L(A_u)$. The dual follows in a similar manner. □

<u>2.10 LEMMA</u>. Let $\{A_u \mid u \in U\}$ be a collection of classes of left R-modules. If the classes A_u are each T-closed, then so is their intersection. Dually, if the classes A_u are each F-closed, then so is their intersection.

Proof: Let the classes A_u each be T-closed. Then $T(\cap A_u) = LR(\cap A_u) = LR(\cap T(A_u)) = LR(\cap LR(A_u)) = LRL(\cup R(A_u)) = L(\cup R(A_u)) = \cap LR(A_u) = \cap T(A_u) = \cap A_u$. Hence $\cap A_u$ is T-closed. The dual follows in a similar manner. □

2.11 THEOREM. Let (T_u, F_u), $u \in U$, be a collection of torsion theories for ${}_RM$. Then $(\cap T_u, F(\cup F_u))$ and $(T(\cup T_u), \cap F_u)$ are also torsion theories for ${}_RM$.

Proof: Since each (T_u, F_u) is a torsion theory, each T_u is T-closed, each F_u is F-closed, $R(T_u) = F_u$, and $L(F_u) = T_u$ by 2.5. Thus by 2.10, $\cap T_u$ is T-closed and $\cap F_u$ is F-closed. Now $R(\cap T_u) = R(\cap L(F_u)) = RL(\cup F_u) = F(\cup F_u)$. Likewise $L(\cap F_u) = L(\cap R(T_u)) = LR(\cup T_u) = T(\cup T_u)$. So by 2.5, $(\cap T_u, F(\cup F_u))$ and $(T(\cup T_u), \cap F_u)$ are also torsion theories for ${}_RM$. □

Thus the torsion-free class corresponding to the torsion class $\cap T_u$ is the smallest torsion-free class containing $\cup F_u$. Also the torsion class corresponding to the torsion-free class $\cap F_u$ is the smallest torsion class containing $\cup T_u$.

CHAPTER III

HEREDITARY TORSION THEORIES AND TORSION FILTERS

3.1 DEFINITION. A set $\mathcal{B}$ of left ideals of R is called a torsion filter provided $\mathcal{B} \neq \emptyset$ and

(a) If $I \in \mathcal{B}$ and if $I \leq I' \leq R$, then $I' \in \mathcal{B}$;

(b) If $I, I' \in \mathcal{B}$, then $I \cap I' \in \mathcal{B}$;

(c) If $I \in \mathcal{B}$, then $(I:a) \in \mathcal{B}$ for all $a \in R$;

(d) If ${}_RI \leq {}_RR$ and if there exists an $I' \in \mathcal{B}$ such that $(I:a) \in \mathcal{B}$ for all $a \in I'$, then $I \in \mathcal{B}$.

It is not difficult to see that (d) implies

(e) If $I, I' \in \mathcal{B}$, then $I \cdot I' \in \mathcal{B}$.

To verify this we have that $(I \cdot I' : a) = \{r \mid ra \in I \cdot I'\} \supseteq I$ for all $a \in I'$, and hence $(I \cdot I' : a) \in \mathcal{B}$ for all $a \in I'$ and by (d) $I \cdot I' \in \mathcal{B}$.

One may observe at this point that the set of essential left ideals of a ring R does not form a torsion filter, because the set of essential left ideals is not closed under products. As an example, consider the ring $Z_4 = \{0,1,2,3\}$. The only essential ideal is $I = \{0,2\}$ and $I \cdot I = \{0\}$ which is not essential in Z_4. However, the set of dense left ideals, a subset of the essential left ideals of a ring R, does form a torsion filter as will be verified later in 5.1.3.

3.2 THEOREM. If T is a hereditary torsion class, then $F(T) = \{{}_R I \leq {}_R R \mid R/I \in T\}$ is a torsion filter.

Proof: Let T be a hereditary torsion class. Clearly $F(T) \neq \emptyset$ since $R \in F(T)$. We now need to show $F(T)$ satisfies (a) - (d) of 3.1.

(a) Let $I \in F(T)$ and let $I \leq I' \leq R$. Consider the exact sequence $R/I \to R/I' \to 0$. Since $I \in F(T)$, then $R/I \in T$, and thus $R/I' \in T$ since T is closed under homomorphic images. Therefore $I' \in F(T)$.

(b) Let $I, I' \in F(T)$, and thus $R/I, R/I' \in T$. By the second isomorphism theorem $I'/I\cap I' \cong I+I'/I \subseteq R/I \in T$. Since T is hereditary, $I+I'/I \in T$, and thus $I'/I\cap I' \in T$. Consider now the exact sequence $0 \to I'/I\cap I' \to R/I\cap I' \to R/I' \to 0$. Since $I/I\cap I' \in T$ and $R/I' \in T$, and since T is a torsion class and so is closed under extensions, then $R/I\cap I' \in T$. Therefore $I \cap I' \in F(T)$.

(c) Let $I \in F(T)$ and $a \in R$. Then $R/I \in T$. Also $Ra+I/I \subseteq R/I$, and since T is hereditary we now have $Ra+I/I \in T$. Consider the exact sequence $R \xrightarrow{f} Ra+I/I \to 0$ where $f(r) = ra + I$ for all $r \in R$. By the first isomorphism theorem $R/\text{Ker } f \cong Ra+I/I$. But $\text{Ker } f = (I:a)$, and so $R/(I:a) \cong Ra+I/I \in T$. Therefore $R/(I:a) \in T$ and $(I:a) \in F(T)$.

(d) Let ${}_R I \leq {}_R R$ and assume there exists $I' \in F(T)$ such that $(I:a) \in F(T)$ for all $a \in I'$. Therefore we now have

$R/I' \in T$ and $R/(I:a) \in T$ for all $a \in I'$. Consider the exact sequence $R/I' \to R/I'+I \to 0$. Since T is closed under homomorphic images, then $R/I'+I \in T$. As in (c), $R/(I:a) \cong Ra+I/I$ for each $a \in I'$, so $Ra+I/I \in T$ for each $a \in I'$. But this implies $\bigoplus \sum_{a \in I'} Ra+I/I \in T$ and hence $\sum_{a \in I'} Ra+I/I \in T$. But $\sum_{a \in I'} Ra+I/I = I'+I\ /I$ so $I'+I/I \in T$. The exact sequence $0 \to I'+I/I \to R/I \to R/I'+I \to 0$ has ends in T, and thus $R/I \in T$ and $I \in F(T)$. □

As an example of 3.2 we shall find the torsion filter for the ring of integers Z associated with the usual torsion theory for abelian groups, which is hereditary. Since every ideal of Z is of the form nZ for some non-negative integer n, and since $Z_n \cong Z/nZ$ is torsion for every positive integer n, we see that the torsion filter for Z is the set of all nonzero ideals of Z.

<u>3.3 THEOREM</u>. If B is a torsion filter, then $T = \{M \in {}_R M \mid (0:x) \in B \text{ for all } x \in M\} = \{M \in {}_R M \mid \text{for all } x \in M, Ix = 0 \text{ for some } I \in B\}$ is a hereditary torsion class.

<u>Proof</u>: Let B be a torsion filter and T as defined above. We will know T is a hereditary torsion class if we can show T is closed under (i) submodules, (ii) homomorphic images, (iii) extensions, and (iv) arbitrary direct sums.

(i) Clearly T is closed under submodules by the way T is defined.

(ii) Let $A \xrightarrow{f} B \to 0$ be an exact sequence where $A \in \mathcal{T}$. Let $b \in B$. Then there exists $a \in A$ such that $f(a) = b$. Since $a \in A$ and $A \in \mathcal{T}$, there exists $I \in \mathcal{B}$ such that $Ia = 0$. Now $Ib = If(a) = f(Ia) = f(0) = 0$, and hence $B \in \mathcal{T}$. Thus $\mathcal{T}$ is closed under homomorphic images.

(iii) Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence with A and C in $\mathcal{T}$. Let $b \in B$ and let $g(b) = c$. Since $c \in C$ and $C \in \mathcal{T}$, then $(0:c) \in \mathcal{B}$. If we can show $((0:b):z) \in \mathcal{B}$ for all $z \in (0:c)$, by 3.1 (d) we will know $(0:b) \in \mathcal{B}$. It is easily seen that $((0:b):z) = (0:zb)$, so our problem is reduced still further to showing $(0:zb) \in \mathcal{B}$ for all $z \in (0:c)$. So let $z \in (0:c)$, and then $z \in (0:c) \Rightarrow zc = 0 \Rightarrow zg(b) = 0 \Rightarrow g(zb) = 0 \Rightarrow$ $zb \in \text{Ker } g = \text{Im } f \Rightarrow$ there exists $a \in A$ such that $f(a) = zb$. One can check $(0:a) \leq (0:zb)$, and by 3.1 (a), since $(0:a) \in \mathcal{B}$, then $(0:zb) \in \mathcal{B}$. This is what we need to get $B \in \mathcal{T}$, and thus $\mathcal{T}$ is closed under extensions.

(iv) Let $\{T_i \mid i \in I\}$ be a collection of elements of $\mathcal{T}$. Let $x \in \oplus \sum_I T_i$. Then for each $i \in I$, $\pi_i x = t_i$ where $t_i \in T_i$ and all but a finite number of the t_i are zero. Let $I_1, I_2, \ldots, I_n$ be the ideals of $\mathcal{B}$ such that $I_i t_i = 0$ for the nonzero t_i. Then by 3.1 (b), $\bigcap_{i=1}^{n} I_i \in \mathcal{B}$ and $(\bigcap_{i=1}^{n} I_i)x = 0$ so $\oplus \sum_I T_i \in \mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary direct sums. □

<u>3.4 THEOREM</u>. There is a one-to-one correspondence between hereditary torsion theories for ${}_R\mathcal{M}$ and torsion filters for R.

Proof: Let ϕ be a function mapping hereditary torsion theories into their associated filters as shown in 3.2. Let ψ be a function mapping torsion filters into their associated hereditary torsion classes as shown in 3.3. We wish to show $\phi\psi$ is the identity on the set of torsion filters for R and that $\psi\phi$ is the identity on the set of hereditary torsion theories for ${}_R\mathcal{M}$.

Let $\mathcal{B}$ be a torsion filter for R. Then

$$\begin{aligned}
\phi\psi(\mathcal{B}) &= \phi(\{M \in {}_R\mathcal{M} \mid (0:x) \in \mathcal{B} \text{ for all } x \in M\}) \\
&= \{{}_RI \le {}_RR \mid R/I \in \{M \in {}_R\mathcal{M} \mid (0:x) \in \mathcal{B} \text{ for all } x \in M\}\} \\
&= \{{}_RI \le {}_RR \mid (0:r+I) \in \mathcal{B} \text{ for all } r \in R\} \\
&= \{{}_RI \le {}_RR \mid (I:r) \in \mathcal{B} \text{ for all } r \in R\} \\
&= \mathcal{B}'.
\end{aligned}$$

We now wish to verify $\mathcal{B} = \mathcal{B}'$. Let $I \in \mathcal{B}$. Then $(I:r) \in \mathcal{B}$ for all $r \in R$ by 3.1 (c). Thus $I \subseteq \mathcal{B}'$ and $\mathcal{B} \subseteq \mathcal{B}'$. Now let $I \in \mathcal{B}'$. Then $(I:r) \in \mathcal{B}$ for all $r \in R$, and hence $I = (I:1) \in \mathcal{B}$. Thus $\mathcal{B}' \subseteq \mathcal{B}$. Therefore $\mathcal{B} = \mathcal{B}'$ and $\phi\psi$ is the identity on the set of torsion filters for R.

Let $\mathcal{T}$ be a hereditary torsion class. Then

$$\begin{aligned}
\psi\phi(\mathcal{T}) &= \psi(\{{}_RI \le {}_RR \mid R/I \in \mathcal{T}\}) \\
&= \{M \in {}_R\mathcal{M} \mid (0:x) \in \{{}_RI \le {}_RR \mid R/I \in \mathcal{T}\} \text{ for all } x \in M\} \\
&= \{M \in {}_R\mathcal{M} \mid R/(0:x) \in \mathcal{T} \text{ for all } x \in M\} \\
&= \mathcal{T}'.
\end{aligned}$$

We wish to verify that $\mathcal{T} = \mathcal{T}'$. Let $M \in \mathcal{T}$ and let $x \in M$. Then $Rx \le M$ and since $\mathcal{T}$ is hereditary $Rx \in \mathcal{T}$. Now $R/(0:x) \cong Rx$

and thus $R/(0:x) \in \mathcal{T}$. Since x was arbitrarily chosen, $R/(0:x) \in \mathcal{T}$ for all $x \in M$, and hence $M \in \mathcal{T}'$. Thus $\mathcal{T} \subseteq \mathcal{T}'$. Now let $M \in \mathcal{T}'$. Then $R/(0:x) \in \mathcal{T}$ for all $x \in M$, and since $R/(0:x) \cong Rx$, then $Rx \in \mathcal{T}$ for all $x \in M$. Therefore $\bigoplus\sum_{x\in M} Rx \in \mathcal{T}$ since $\mathcal{T}$ is closed under direct sums, and $\sum_{x\in M} Rx \in \mathcal{T}$ since $\mathcal{T}$ is closed under homomorphic images. But $M = \sum_{x\in M} Rx$, and thus $M \in \mathcal{T}$ and $\mathcal{T}' \subseteq \mathcal{T}$. We now have $\mathcal{T} = \mathcal{T}'$, and hence $\psi\phi$ is the identity on the set of hereditary torsion theories for ${}_R M$. □

Since a torsion filter $\mathcal{B}$ uniquely determines a set of cyclic modules $\{R/I \mid I \in \mathcal{B}\}$, then 3.4 shows that a hereditary torsion class is uniquely determined by the cyclic modules in it.

<u>3.5 THEOREM</u>. Let $\mathcal{T}$ and $\mathcal{T}'$ be a hereditary torsion theories for ${}_R M$. Then $F(\mathcal{T}) \subseteq F(\mathcal{T}')$ if and only if $\mathcal{T} \subseteq \mathcal{T}'$.

<u>Proof</u>: ($\rightarrow$) Let $F(\mathcal{T}) \subseteq F(\mathcal{T}')$. Let $M \in \mathcal{T}$ and $m \in M$. Then $R/(0:m) \cong Rm \leq M \in \mathcal{T}$, and since $\mathcal{T}$ is hereditary, $Rm \in \mathcal{T}$. Therefore $R/(0:m) \in \mathcal{T}$ and $(0:m) \in F(\mathcal{T})$. Since $F(\mathcal{T}) \subseteq F(\mathcal{T}')$, then $(0:m) \in F(\mathcal{T}')$, $R/(0:m) \in \mathcal{T}'$, and $Rm \in \mathcal{T}'$. This is true for each $m \in M$, and hence $\bigoplus\sum_{m\in M} Rm \in \mathcal{T}'$. Therefore $M = \sum_{m\in M} Rm \in \mathcal{T}'$, and we have $\mathcal{T} \subseteq \mathcal{T}'$.

($\leftarrow$) Let $\mathcal{T} \subseteq \mathcal{T}'$. Then $I \in F(\mathcal{T}) \Rightarrow R/I \in \mathcal{T} \subseteq \mathcal{T}' \Rightarrow I \in F(\mathcal{T}')$. Therefore $F(\mathcal{T}) \subseteq F(\mathcal{T}')$. □

CHAPTER IV

TTF CLASSES

4.1 DEFINITION. Let $\mathcal{T}$ be a class of modules in ${}_R\mathcal{M}$ which is closed under submodules, homomorphic images, extensions, arbitrary direct sums, and arbitrary direct products. Then $\mathcal{T}$ is the torsion class for a torsion theory $(\mathcal{T},\mathcal{F})$ which is hereditary, and $\mathcal{T}$ is also the torsion-free class for another torsion theory $(\mathcal{C},\mathcal{T})$. Such a pair of torsion theories $(\mathcal{C},\mathcal{T})$ and $(\mathcal{T},\mathcal{F})$ will be called a torsion-torsion-free theory (TTF theory) and the class $\mathcal{T}$ will be called a TTF class.

With this definition of a TTF class, one immediately observes that a hereditary torsion class $\mathcal{T}$ misses being a TTF class by the property of being closed under arbitrary direct products. One then begins to look for a condition that will make $\mathcal{T}$ into a TTF class. The following theorem reveals an answer.

4.2 THEOREM. Let $\mathcal{T}$ be a hereditary torsion class. Then $\mathcal{T}$ is a TTF class if and only if its associated filter $F(\mathcal{T})$ has a smallest element.

Proof: $(\rightarrow)$ Let $\mathcal{T}$ be a TTF class and consider $\Pi\{R/I \mid I \in F(\mathcal{T})\}$, which is in $\mathcal{T}$ since $\mathcal{T}$ is closed under direct products. Let $f \in \mathrm{Hom}(R,\Pi\{R/I \mid I \in F(\mathcal{T})\})$ be defined by

$\pi_I\, f(r) = r + I$ for all $I \in F(T)$ and $r \in R$. Then $\operatorname{Im} f \in T$ since T is closed under submodules. But $R/\operatorname{Ker} f \cong \operatorname{Im} f$, and hence $R/\operatorname{Ker} f \in T$ and $\operatorname{Ker} f \in F(T)$. Also $\operatorname{Ker} f \subseteq I$ for each $I \in F(T)$ and thus $\operatorname{Ker} f = \cap\{I \mid I \in F(T)\}$ and is the smallest element of $F(T)$.

($\Leftarrow$) Now let T be a hereditary torsion class and assume $F(T)$ has a smallest element I. We first wish to show $T = \{M \in {}_R M \mid IM = 0\}$. Let $M \in T$ and $x \in M$. Then $(0:x) \cap I \in F(T)$, so $I = (0:x) \cap I$ and $Ix = 0$. Therefore $IM = 0$, $M \in \{M \in {}_R M \mid IM = 0\}$, and $T \subseteq \{M \in {}_R M \mid IM = 0\}$. Now let $M \in {}_R M$ such that $IM = 0$. Let $x \in M$. Then $R/(0:x) \simeq Rx$ and since $IM = 0$, $I \subseteq (0:x)$. Hence $(0:x) \in F(T)$ and $R/(0:x) \in T$. Therefore $Rx \in T$ for all $x \in M$. Thus $\bigoplus \sum_{x \in M} Rx \in T$ and $M = \sum_{x \in M} Rx \in T$. So we have $\{M \in {}_R M \mid IM = 0\} \subseteq T$. We have then shown $T = \{M \in {}_R M \mid IM = 0\}$. Since T is a hereditary torsion class, it remains to show that T is closed under direct products in order to verify T is a TTF class. Let $\{T_i \mid i \in I\}$ be a collection of elements of T. Then $I \cdot \prod_I T_i \cong \prod_I I \cdot T_i \cong 0$ and hence $\prod_I T_i \in T$ and T is closed under direct products. $\square$

We would now like to observe that if T is a TTF class with (C,T) and (T,F) its associated torsion theories, then the smallest element I of $F(T)$ turns out to be R_c, the C-torsion submodule of ${}_R R$ in the torsion theory (C,T). By definition of a torsion theory we have that for ${}_R R$ there exists a unique submodule R_c of ${}_R R$ such that $R_c \in \mathcal{C}$ and $R/R_c \in \mathcal{T}$. Hence $R_c \in F(\mathcal{T})$ which implies $I \subseteq R_c$ since I is the smallest element of $F(\mathcal{T})$. Now the exact sequence $R_c \to R_c/I \to 0$ gives $R_c/I \in \mathcal{C}$. Also $R_c/I \subseteq R/I \in \mathcal{T}$, so $R_c/I \in \mathcal{T}$. Therefore $R_c/I \in \mathcal{C} \cap \mathcal{T} = 0$ and hence $R_c = I$.

We next observe that R_t(and R_c) are each two-sided ideals of R. Obviously R_t is a left ideal since R_t is a left R-module. We now wish to show $R_t R \subseteq R_t$. Let $a \in R$ and consider the exact sequence $R_t \xrightarrow{f} R_t a \to 0$ where $f(x) = xa$ for all $x \in R_t$. Then $R_t a \in \mathcal{T}$, and since a was arbitrarily chosen, $R_t a \in \mathcal{T}$ for all $a \in R$ and $R_t R \subseteq \mathcal{T}$. Since R_t is the largest submodule of R contained in $\mathcal{T}$ then $R_t R \subseteq R_t$. Therefore R_t is a two-sided ideal, and similarly R_c is a two-sided ideal.

It is also interesting to note here that if R is a left Artinian ring, then every hereditary torsion class is a TTF class since the filter always has a smallest element.

We conclude our discussion of TTF classes with the following theorem which gives information about a special class of TTF classes.

<u>4.3 THEOREM</u>. Let $\mathcal{T}$ be a TTF class, and let $(\mathcal{T},\mathcal{F})$ and $(\mathcal{C},\mathcal{T})$ be the torsion theories associated with $\mathcal{T}$. Then the following are equivalent:

(1) $M = M_c \oplus M_t$ for all modules M;

(2) $R = R_c \oplus R_t$ (ring direct sum);

(3) $\mathcal{F} = \mathcal{C}$;

(4) $(M_c)_t = 0$ and $(M/M_t)_c = M/M_t$ for all modules M;

(5) $\mathcal{T}$ is closed under injective envelopes and R_c is a direct summand of R;

(6) $\mathcal{F}$ is closed under homomorphic images and R_t is a direct summand of R;

(7) R_c is a ring direct summand of R.

Proof: (1→2) Let $M = M_c \oplus M_t$ for all modules M. Then clearly $R = R_c \oplus R_t$ where R is a module over itself. Since R_t and R_c are two sided ideals, we then have $R = R_c \oplus R_t$ is a ring direct sum.

(2→3) Let $R = R_c \oplus R_t$ be a ring direct sum. Before attempting to prove $\mathcal{F} = \mathcal{C}$, we would like to verify that since $R = R_c \oplus R_t$ and since R_c and R_t are two sided ideals, then $1 = e_1 + e_2$, where e_1 and e_2 are orthogonal central idempotents in R, $R_c = Re_1$, and $R_t = Re_2$. First we see that $e_1e_1 = e_1(1-e_2) = e_1 - e_1e_2$, so $e_1 - e_1e_1 = e_1e_2 \in R_c \cap R_t = 0$ and thus $e_1 = e_1e_1$ and e_1 is idempotent. Likewise e_2 is idempotent. Also $e_1e_2 = e_1(1-e_1) = e_1 - e_1e_1 = e_1 - e_1 = 0$, and so e_1 and e_2 are orthogonal. Let $x \in R$ and try to show $e_1x = xe_1$. Since R_c is a two sided ideal, $e_1x - xe_1 \in R_c$. Also $e_1x - xe_1 = (1-e_2)x - x(1-e_2) = x - e_2x - x + xe_2 = xe_2 - e_2x \in R_t$ and so $e_1x - xe_1 \in R_c \cap R_t = 0$. Thus $e_1x = xe_1$ which implies e_1 is in the centralizer of R. Similarly e_2 is in the centralizer of R. Thus e_1 and e_2 are orthogonal central idempotents in R. To observe that $R_c = Re_1$, let $x \in R_c$ and then $x = x \cdot 1 = x(e_1+e_2) = xe_1 + xe_2$. So $x - xe_1 = xe_2 \in R_c \cap R_t = 0$, $x = xe_1$, $x \in Re_1$, and $R_c \subseteq Re_1$. We already know $Re_1 \subseteq R_c$, and thus $R_c = Re_1$. Similarly $R_t = Re_2$. Now we return to the original problem of showing $\mathcal{F} = \mathcal{C}$. From proof of 4.2 we already know $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid R_cM = 0\}$. Let $\mathcal{A} = \{M \in {}_R\mathcal{M} \mid R_tM = 0\}$. Since $1 = e_1 + e_2$ where e_1 and e_2 are orthogonal central idempotents in R we see that $\mathcal{T} = \{M \in {}_R\mathcal{M} \mid e_1M = 0\}$ and $\mathcal{A} = \{M \in {}_R\mathcal{M} \mid e_2M = 0\}$. One should also observe here that $e_2m = m$ for all $m \in M$ where $M \in \mathcal{T}$, and $e_1m = m$ for all $m \in M$ where $M \in \mathcal{A}$. We now claim that (i) $\mathcal{A} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,T) = 0$ for all $T \in \mathcal{T}\}$ and (ii) $\mathcal{A} = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(T,M) = 0$ for all $T \in \mathcal{T}\}$. To verify (i), let $M \in \mathcal{A}$. Let $T \in \mathcal{T}$ and $f \in \mathrm{Hom}(M,T)$. Assume $f \neq 0$. Then $\mathrm{Im}\, f \neq 0$ so there must exist $0 \neq m \in M$ such that $f(m) \neq 0$. Now $e_2M = 0$, so $e_2m = 0$ and hence $0 = f(e_2m) = e_2f(m) = f(m) \neq 0$ and this is a contradiction. Thus $f = 0$ and we have $\mathcal{A} \subseteq \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,T) = 0$ for all $T \in \mathcal{T}\}$. Now let M be a module such that $\mathrm{Hom}(M,T) = 0$ for all $T \in \mathcal{T}$. Then $e_1(e_2M) = 0$ so $e_2M \in \mathcal{T}$. Let $f \in \mathrm{Hom}(M,e_2M)$ defined by $f(m) = e_2m$ for all $m \in M$. Since $f = 0$, we have $\mathrm{Im}\, f = 0$, $e_2M = 0$, and $M \in \mathcal{A}$. Thus $\{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,T) = 0$ for all $T \in \mathcal{T}\} \subseteq \mathcal{A}$. Therefore we have verified (i) and similarly we could verify (ii). But if $\mathcal{A}$ is characterized as in (i) and (ii), then we immediately have $\mathcal{F} = \mathcal{A} = \mathcal{C}$.

(3→4) Let $F = C$ and let M be a module. We quickly have $(M_c)_t \leq M_c \in C$ and $(M_c)_t \in T$. Since $F = C$ we know C is closed under submodules, and hence $(M_c)_t \in C$. Thus $(M_c)_t \in C \cap T = 0$. We also know $M/M_t \in F = C$ and hence $(M/M_t)_c = M/M_t$.

(4→1) Let $(M_c)_t = 0$ and $(M/M_t)_c = M/M_t$ for all modules M and let M be a module. Consider first $M_c \cap M_t$. Then $M_c \cap M_t \subseteq M_t \in T$. Also $M_c \cap M_t \subseteq M_c$, but since $(M_c)_t = 0$ then M_c has no nonzero T-torsion submodules. But $M_c \cap M_t$ is a T-torsion submodule of M_c and hence $M_c \cap M_t = 0$. Now consider M/M_c+M_t which is a factor of $M/M_c \in T$ and hence $M/M_c+M_t \in T$. But M/M_c+M_t is also a factor of $M/M_t = (M/M_t)_c \in C$ and hence $M/M_c+M_t \in C$. Therefore $M/M_c+M_t \in C \cap T = 0$, $M = M_c + M_t$, and thus $M = M_c \oplus M_t$.

(2 and 3→5,6, and 7) The proof is immediate.

(5→2) Let T be closed under injective envelopes and R_c a direct summand of R. Thus C is hereditary by 1.8 and $R = R_c \oplus I$. Then $I \cong R/R_c \in T$ and thus $I \subseteq R_t$. So $R = R_c + R_t$, but $R_c \cap R_t \in C \cap T = 0$ and $R = R_c \oplus R_t$.

(6→2) Let F be closed under homomorphic images and R_t a direct summand of R. We have $R = R_t \oplus I$ and $I \cong R/R_t \in F$. Also $R/I \cong R_t \in T$ so $I \in F(T)$, and $R_c \subseteq I$ since R_c is the smallest element of the filter. Now consider the exact sequence

$I \to I/R_c \to 0$ with I in F, so $I/R_c \in F$. The sequence $0 \to I/R_c \to R/R_c$ is also exact with R/R_c in T, so $I/R_c \in T$. Now we have $I/R_c \in T \cap F = 0$ and $I = R_c$. Therefore $R = R_t \oplus R_c$.

(7→2) Let R_c be a ring direct summand of R. Then $R = R_c \oplus I$ where I is a two-sided ideal of R. Then $I \cong R/R_c \in T$ so $I \subseteq R_t$ and $R = R_c + R_t$. As verified in (2→3) $R_c = Re$ where $e \in R_c$ and e is a central idempotent. Let $x \in R_c \cap R_t$. Then $x = ae$ where $a \in R$, and $x \in R_t$ so $(0:x) \in F(T)$ by 3.3. Since R_c is the smallest element of the filter, $R_c \subseteq (0:x)$ and $R_c x = 0$. Thus $ex = 0$, so that $0 = ex = eae = aee = ae = x$ and $R_c \cap R_t = 0$. Hence $R = R_c \oplus R_t$. □

CHAPTER V

SPECIFIC TORSION THEORIES

We now go about the task of constructing several specific torsion theories for ${}_R\mathcal{M}$, namely the E(R)-torsion theory, the Goldie torsion theory, and the simple torsion theory. Each construction will be valid for an arbitrary ring R, and each will yield a hereditary torsion class. Our goal will be to compute the filters for each of these torsion classes, to investigate when each torsion theory coincides with the standard concept of torsion for a module over an integral domain, and also to investigate inclusion relationships among these torsion classes.

SECTION 1: E(R)-Torsion Theory

Let R be an arbitrary ring and let E(R) be the injective hull of R considered as a left module over itself. Let $\mathcal{T}_0 = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M,E(R)) = 0\}$. From 2.1 we see that $\mathcal{T}_0$ is a hereditary torsion class. The torsion theory associated with the torsion class $\mathcal{T}_0$ is called the <u>E(R)-torsion theory</u> and modules in $\mathcal{T}_0$ are said to be <u>E(R)-torsion</u>.

The following theorem yields an alternate way of defining the E(R)-torsion theory.

<u>5.1.1 THEOREM</u>. Let $\mathcal{T}_0' = \{M \in {}_R\mathcal{M} \mid \mathrm{Hom}(M',R) = 0$ for all $M' \leq M\}$. Then $\mathcal{T}_0' = \mathcal{T}_0$.

Proof: Let $M \in \mathcal{T}_0'$ and let $f \in \mathrm{Hom}(M,E(R))$. Assume $f \neq 0$ and thus $\mathrm{Im}\, f \neq 0$. But since $R \trianglelefteq E(R)$ we have $\mathrm{Im}\, f \cap R \neq 0$. Let $M' = f^{-1}(R \cap \mathrm{Im}\, f)$ and define $f' : M' \to R$ by $f'(x) = f(x)$ for all $x \in M'$. But then we have $f' \neq 0$ and this is a contradiction to the fact $M \in \mathcal{T}_0'$. Hence $f = 0$, $M \in \mathcal{T}_0$, and $\mathcal{T}_0' \subseteq \mathcal{T}_0$.

Now let $M \in \mathcal{T}_0$, let $M' \leq M$, and let $f \in \mathrm{Hom}(M',R)$. Consider the following diagram with exact row:

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "i_{M'}"] \arrow[d, "f"'] & M \arrow[ddl, dashed, "h"] \\ & R \arrow[d, "i_R"'] & \\ & E(R) & \end{tikzcd}$$

Since $E(R)$ is injective, there exists an $h \in \mathrm{Hom}(M,E(R))$ such that the diagram commutes; that is, $hi_{M'} = i_R f$. But since $h \in \mathrm{Hom}(M,E(R))$, then $h = 0$ and hence $f = 0$. Thus $M \in \mathcal{T}_0'$ and $\mathcal{T}_0 \subseteq \mathcal{T}_0'$. Therefore $\mathcal{T}_0' = \mathcal{T}_0$. □

5.1.2 THEOREM. If $\mathcal{T}$ is a hereditary torsion class and R is $\mathcal{T}$-torsion-free, then $\mathcal{T} \subseteq \mathcal{T}_0$.

Proof: Let $\mathcal{T}$ be a hereditary torsion class and let R be $\mathcal{T}$-torsion-free. Let $T \in \mathcal{T}$. Then since $\mathcal{T}$ is hereditary we know that every submodule of T is in $\mathcal{T}$. Since R is $\mathcal{T}$-torsion-free, $\mathrm{Hom}(T',R) = 0$ for all $T' \leq T$, and $T \in \mathcal{T}_0$ by 5.1.1. Therefore $\mathcal{T} \subseteq \mathcal{T}_0$. □

As a consequence of 5.1.2, we have that the E(R)-torsion theory is the largest hereditary torsion theory for which the ring is torsion free.

<u>5.1.3 THEOREM</u>. Let L be a left ideal of R. Then $L \in F(\mathcal{T}_0)$, the filter associated with the E(R)-torsion class $\mathcal{T}_0$, if and only if L is dense.

<u>Proof</u>: ($\rightarrow$) Let $L \in F(\mathcal{T}_0)$. Let $a,b \in R$ such that $(L:a) \cdot b = 0$. Note here that $L \in F(\mathcal{T}_0) \Rightarrow R/L \in \mathcal{T}_0 \Rightarrow \mathrm{Hom}(R'/L, R) = 0$ for all $R'/L \leq R/L$ by 5.1.1. Define $f : Ra + L/L \rightarrow R$ by $f(ra+L) = rb$. One can easily check f is a well-defined R-homomorphism from a submodule of R/L into R, and hence $f = 0$. Thus $Rb = \mathrm{Im}\, f = 0$ and hence $b = 1 \cdot b = 0$. Thus $L \in \mathcal{D}$, the set of dense left ideals of R, and $F(\mathcal{T}_0) \subseteq \mathcal{D}$.

Now let $L \in \mathcal{D}$. Let $f \in \mathrm{Hom}(R/L, E(R))$. Assume $f \neq 0$. Thus $\mathrm{Im}\, f \neq 0$ and $\mathrm{Im}\, f \cap R \neq 0$. There exists $0 \neq y \in \mathrm{Im}\, f$ and $y \in R$. Thus there is $0 \neq x + L \in R/L$ such that $f(x+L) = y$. We would now like to verify $(L:x) \cdot y = 0$. Let $r \in (L:x)$. Then $ry = rf(x+L) = f(rx+L) = f(L) = 0$. Thus $(L:x) \cdot y = 0$ which implies $y = 0$ since $L \in \mathcal{D}$. This is a contradiction, and thus $f = 0$, $R/L \in \mathcal{T}_0$, and $L \in F(\mathcal{T}_0)$. Therefore $\mathcal{D} \subseteq F(\mathcal{T}_0)$. We have now verified $F(\mathcal{T}_0) = \mathcal{D}$. □

Applying 5.1.3, 5.1.2, and 3.5 we can now observe that the set of dense left ideals forms the maximum torsion filter for a ring which is torsion-free.

5.1.4 THEOREM. The E(R)-torsion theory coincides with the standard concepts of torsion and torsion-free for modules over an integral domain.

Proof: Let R be an integral domain and let M be a left R-module. Let $T(M) = \{x \in M \mid$ there exists $0 \neq r \in R$ such that $rx = 0\}$. Let M_0 be the torsion submodule of M in the E(R)-torsion theory $\mathcal{T}_0$ and hence $M_0 = \{x \in M \mid (0:x) \in F(\mathcal{T}_0)\}$. We now wish to verify $T(M) = M_0$. Let $x \in T(M)$. Then there exists $0 \neq r \in R$ such that $rx = 0$. Let $a,b \in R$ such that $((0:x):a) \cdot b = 0$, and hence $(0:ax) \cdot b = 0$. Then $r \in (0:ax)$ since $rax = arx = a0 = 0$ and we have $rb = 0$. Since $r \neq 0$ and R is an integral domain, then $b = 0$. Thus $(0:x)$ is dense, $(0:x) \in F(\mathcal{T}_0)$, $x \in M_0$, and $T(M) \subseteq M_0$. Now let $x \in M_0$. Assume $(0:x) = 0$. Then $0 = (0:x) \in F(\mathcal{T}_0)$ which implies $R = R/0 \in \mathcal{T}_0$. This is a contradiction because $R \in \mathcal{F}_0$. Hence there is a $0 \neq r \in R$ such that $r \in (0:x)$, and hence $rx = 0$, $x \in T(M)$, and $M_0 \subseteq T(M)$. Therefore $T(M) = M_0$ and the E(R)-torsion theory coincides with the standard concept of torsion for a module over an integral domain. □

SECTION 2: Goldie Torsion Theory

Before introducing the Goldie torsion theory, we dispense with some needed results.

<u>5.2.1 LEMMA</u>. Let L be an essential left ideal of R and let $x \in R$. Then $(L:x)$ is an essential left ideal of R.

<u>Proof</u>: Suppose $(L:x) \cap K = 0$ for some $K \leq R$. Then $L \cap Kx = 0$, and since $L \trianglelefteq R$, we have $Kx = 0$. But $Kx = 0$ implies $K \subseteq (L:x)$ and this in turn implies $K = 0$. Thus $(L:x) \trianglelefteq R$. □

<u>5.2.2 LEMMA</u>. For any module M, $Z(E(M)/M) = E(M)/M$.

<u>Proof</u>: It remains to show $E(M)/M \subseteq Z(E(M)/M)$. Let $e + M \in E(M)/M$. We must show $(0:e+M) = (M:e) \trianglelefteq R$. Let $0 \neq a \in R$. Then $ae \in E(M)$ and so $Rae \leq E(M)$. Hence $Rae \cap M \neq 0$. There exists $r \in R$ such that $rae \neq 0$ and $rae \in M$. Thus $ra \neq 0$ and $Ra \cap (M:e) \neq 0$ and $(M:e) \trianglelefteq R$. Thus $e + M \in Z(E(M)/M)$ and $Z(E(M)/M) = E(M)/M$. □

Recall that we define $Z_i(M) = \{x \in M \mid (Z_{i-1}(M):x) \trianglelefteq R\}$ for $i = 2,3,4,\ldots$ where $Z_1(M) = Z(M)$. In general $Z_1(M) \subsetneq Z_2(M) \subsetneq M$.

<u>5.2.3 LEMMA</u>. Let M be a module. Then $Z_2(M) = Z_3(M)$.

<u>Proof</u>: By definition we already have $Z_2(M) \subseteq Z_3(M)$. Now let $m \in Z_3(M)$. Then $(Z_2(M):m) \trianglelefteq R$ and we want to show $(Z_1(M):m) \trianglelefteq R$. Let $0 \neq x \in R$. Then $Rx \cap (Z_2(M):m) \neq 0$. Then there exists $r \in R$ such that $rx \neq 0$ and $rxm \in Z_2(M)$. If $rxm \in Z_1(M)$ we are through. If $rxm \notin Z_1(M)$, then $(0:rxm) \ntrianglelefteq R$.

Thus there exists $K \leq R$ such that $K \neq 0$ and $K \cap (0{:}rxm) = 0$. Let $0 \neq y \in K$. Since $rxm \in Z_2(M)$ we know $(Z_1(M){:}rxm) \trianglelefteq R$ and thus $Ry \cap (Z_1(M){:}rxm) \neq 0$. Thus there exists $r' \in R$ with $r'y \neq 0$ and $r'yrxm \in Z_1(M)$. Then $r'y$ is a nonzero element of K and hence $r'yrxm \neq 0$ since $K \cap (0{:}rxm) = 0$. Thus $r'yrx \neq 0$ and $Rx \cap (Z_1(M){:}m) \neq 0$ and $(Z_1(M){:}m) \trianglelefteq R$. Then $m \in Z_2(M)$ and $Z_3(M) \subseteq Z_2(M)$. Therefore $Z_2(M) = Z_3(M)$. □

<u>5.2.4 LEMMA</u>. For a module M, $Z_2(Z_2(M)) = Z_2(M)$.

<u>Proof</u>: Let M be a module. Then by definition, $Z_2(Z_2(M)) \subseteq Z_2(M)$. Now let $m \in Z_2(M)$. Then $(Z_1(M){:}m) \trianglelefteq R$. Let $0 \neq x \in R$. Then $Rx \cap (Z_1(M){:}m) \neq 0$. There exists $r \in R$ such that $rx \neq 0$ and $rxm \in Z_1(M)$. But this implies $(0{:}rxm) \trianglelefteq R$. Then $rxm \in Z_1(Z_2(M))$ and $Rx \cap (Z_1(Z_2(M)){:}m) \neq 0$ and $(Z_1(Z_2(M)){:}m) \trianglelefteq R$. Thus $m \in Z_2(Z_2(M))$ and $Z_2(M) \subseteq Z_2(Z_2(M))$. We have now shown $Z_2(Z_2(M)) = Z_2(M)$. □

Now we are ready to start the Goldie torsion theory. Let R be an arbitrary ring and let $\mathcal{G} = \{M \in {}_R\mathcal{M} \mid M \cong R/L$ where L is an essential left ideal of $R\}$.

<u>5.2.5 LEMMA</u>. $\mathcal{G}$ is closed under homomorphic images and cyclic submodules.

<u>Proof</u>: Let $M \in \mathcal{G}$ and $M \to N \to 0$ be an exact sequence. Then $M \cong R/L$ where $L \trianglelefteq R$ and thus $N \cong R/L'$ where $L' \leq R$ and

$L \le L'$. Since $L \trianglelefteq R$ then $L' \trianglelefteq R$ and hence $N \in \mathcal{G}$. Therefore $\mathcal{G}$ is closed under homomorphic images.

Now let $M \in \mathcal{G}$ and let Rx be a cyclic submodule of M where $x \in M$. Since $M \cong R/L$ where $L \trianglelefteq R$, then $Rx \cong Ra + L/L \cong R/(L:a)$ where $a \in R$. Since $(L:a) \trianglelefteq R$ by 5.2.1, we now have $R/(L:a) \in \mathcal{G}$ which implies $Rx \in \mathcal{G}$, and $\mathcal{G}$ is closed under cyclic submodules. □

Since $\mathcal{G}$ is closed under homomorphic images and cyclic submodules, then by 2.8 we have that $\mathcal{T}_\mathcal{G}$ is a hereditary torsion class, and by 2.7 we have $\mathcal{T}_\mathcal{G} = LR(\mathcal{G})$. We call the torsion theory associated with $\mathcal{T}_\mathcal{G}$ the <u>Goldie torsion theory</u>, and modules in $\mathcal{T}_\mathcal{G}$ are said to be <u>Goldie torsion</u>.

<u>5.2.6 LEMMA</u>. For any module M, $Z(M) \in \mathcal{T}_\mathcal{G}$.

<u>Proof</u>: Let M be a module and let $m \in Z(M)$. Then $(0:m) \trianglelefteq R$, and so we have $Rm \cong R/(0:m) \in \mathcal{T}_\mathcal{G}$. Since $\mathcal{T}_\mathcal{G}$ is a torsion class we now have $\bigoplus\sum_{m \in Z(M)} Rm \in \mathcal{T}_\mathcal{G}$ which implies $\sum_{m \in Z(M)} Rm \in \mathcal{T}_\mathcal{G}$. But $Z(M) = \sum_{m \in Z(M)} Rm$ and hence $Z(M) \in \mathcal{T}_\mathcal{G}$. □

<u>5.2.7 THEOREM</u>. The Goldie torsion theory $\mathcal{T}_\mathcal{G}$ is closed under injective envelopes.

<u>Proof</u>: Let $M \in \mathcal{T}_\mathcal{G}$ and let $E(M)$ be the injective hull of M. By 5.2.2 and 5.2.6, $E(M)/M = Z(E(M)/M) \in \mathcal{T}_\mathcal{G}$. We now have the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$ with ends in $\mathcal{T}_\mathcal{G}$ and

hence $E(M) \in \mathcal{T}_G$ since $\mathcal{T}_G$ is closed under extensions. Therefore $\mathcal{T}_G$ is closed under injective envelopes. □

5.2.8 LEMMA. The Goldie torsion theory $\mathcal{T}_G = \{M \in {}_R\mathcal{M} \mid Z_2(M) = M\}$.

Proof: Let $M \in \mathcal{T}_G$. Assume $Z_2(M) \neq M$. Then $M' = M/Z_2(M)$ is a nonzero homomorphic image of M and hence must contain a nonzero submodule $N' \in \mathcal{G}$ and $N' \cong R/L$ where $L \trianglelefteq R$. Let $0 \neq n \in N'$. Then $(0:n) \trianglelefteq R$ and hence $Z(M') \neq 0$. So $0 \neq Z(M') = Z(M/Z_2(M)) = Z_3(M)/Z_2(M)$ and thus $Z_2(M) \neq Z_3(M)$. This contradicts 5.2.3, and hence $Z_2(M) = M$ and we have shown $\mathcal{T}_G \subseteq \{M \in {}_R\mathcal{M} \mid Z_2(M) = M\}$.

Now let $M \in {}_R\mathcal{M}$ such that $Z_2(M) = M$. Hence $M/Z_1(M) = Z_2(M)/Z_1(M) = Z(M/Z_1(M)) \in \mathcal{T}_G$ by 5.2.6. We have the exact sequence $0 \to Z_1(M) \to M \to M/Z_1(M) \to 0$ with both ends in $\mathcal{T}_G$ by 5.2.6, and hence $M \in \mathcal{T}_G$. Thus $\{M \in {}_R\mathcal{M} \mid Z_2(M) = M\} \subseteq \mathcal{T}_G$. Therefore we have shown $\mathcal{T}_G = \{M \in {}_R\mathcal{M} \mid Z_2(M) = M\}$. □

5.2.9 THEOREM. Let $M \in {}_R\mathcal{M}$. Then $M_g = Z_2(M)$ where M_g is the Goldie torsion submodule of M. If $Z(R) = 0$, then $M_g = Z(M)$.

Proof: Since $Z_2(Z_2(M)) = Z_2(M)$ by 5.2.4, then $Z_2(M) \in \mathcal{T}_G$ by 5.2.8. But M_g is the largest Goldie torsion submodule of M and thus $Z_2(M) \subseteq M_g$. Since $M_g \in \mathcal{T}_G$, we have

$M_g = Z_2(M_g)$ by 5.2.8, and also $Z_2(M_g) \subseteq Z_2(M)$. Thus $M_g \subseteq Z_2(M)$. We have thus verified $M_g = Z_2(M)$.

Now let $Z(R) = 0$. We wish to show $Z_1(M) = Z_2(M)$, and then by the first part of the theorem we will have our desired result $M_g = Z(M)$. We already know $Z_1(M) \subseteq Z_2(M)$. Now let $m \in Z_2(M)$. Then $(Z_1(M):m) \trianglelefteq R$. Let $0 \neq x \in R$. Then $Rx \cap (Z_1(M):m) \neq 0$. There exists $r \in R$ such that $rx \neq 0$ and $rxm \in Z_1(M)$. Then we have $(0:rxm) \trianglelefteq R$, and since $Z(R) = 0$, this implies $rxm = 0$. Thus $Rx \cap (0:m) \neq 0$, $(0:m) \trianglelefteq R$, and $m \in Z_1(M)$. Therefore $Z_2(M) \subseteq Z_1(M)$ and we have verified $Z_1(M) = Z_2(M)$. □

The following theorem will tell us how to compute the filter $F(\mathcal{T}_G)$ for the Goldie torsion class $\mathcal{T}_G$ and shows that this filter contains all the essential ideals of R.

5.2.10 THEOREM. Let L be a left ideal of R. Then $L \in F(\mathcal{T}_G)$, the filter associated with the Goldie torsion class $\mathcal{T}_G$, if and only if there exists an essential left ideal L' of R with $L \subseteq L'$ and $(L:x) \trianglelefteq R$ for all $x \in L'$.

Proof: $(\rightarrow)$ Let $L \in F(\mathcal{T}_G)$. Define L' by $L'/L = Z(R/L)$. Clearly $L \subseteq L' \leq R$ and if $x \in L'$, then $(L:x) \trianglelefteq R$. We need to verify $L' \trianglelefteq R$. Since $L \in F(\mathcal{T}_G)$, then $R/L \in \mathcal{T}_G$ and hence $Z_2(R/L) = R/L$ by 5.2.8. By definition $Z_2(R/L)/Z_1(R/L) = Z(R/L/Z_1(R/L))$ which implies $R/L' = Z(R/L')$. We now know $(L':r) \trianglelefteq R$ for all $r \in R$. But $L' = (L':1)$ and so $L' \trianglelefteq R$.

($\leftarrow$) Now assume there exists an essential left ideal L' of R with $L \subseteq L'$ and $(L:x) \trianglelefteq R$ for all $x \in L'$. Then $L'/L = Z(R/L) \in T_G$ by 5.2.6. Also since $L' \trianglelefteq R$, $R/L' \in T_G$. Thus the exact sequence $0 \to L'/L \to R/L \to R/L' \to 0$ has both ends in T_G and we then have $R/L \in T_G$. Therefore $L \in F(T_G)$. □

We remark that if L is an essential left ideal of R, then so is $(L:r)$ for all $r \in R$ and hence $L \in F(T_G)$.

<u>5.2.11 LEMMA</u>. Every dense left ideal of R is essential in R. If $Z(R) = 0$, then every essential left ideal of R is dense.

<u>Proof</u>: Let I be a dense left ideal of R. Let $0 \neq x \in R$. Then $(I:x)x \neq 0$ and so there exists $r \in (I:x)$ such that $rx \neq 0$. Thus $Rx \cap I \neq 0$ and $I \trianglelefteq R$.

Let $Z(R) = 0$ and let I be an essential left ideal of R. Let $a,b \in R$ such that $(I:a) \cdot b = 0$. Let $0 \neq x \in R$. By 5.2.1 $(I:a) \trianglelefteq R$, and hence $Rx \cap (I:a) \neq 0$. There exists $r \in R$ such that $rx \neq 0$ and $rx \in (I:a)$. Hence $rxb = 0$ and $rx \in (0:b)$. Hence $Rx \cap (0:b) \neq 0$ and $(0:b) \trianglelefteq R$, and since $Z(R) = 0$, we have $b = 0$. Therefore I is dense. □

The following theorem will give the inclusion relationships between the Goldie torsion class T_G and the E(R)-torsion class T_0.

<u>5.2.12 THEOREM</u>. The E(R)-torsion class T_0 is contained in the Goldie torsion class T_G, and $Z(R) = 0$ if and only if $T_0 = T_G$.

Proof: If we can show $F(T_0) \subseteq F(T_G)$, then by 3.5 we will have $T_0 \subseteq T_G$. So let $I \in F(T_0)$. Then I is a dense ideal by 5.1.3 and hence I is essential by 5.2.11. Thus $I \in F(T_G)$ since $F(T_G)$ contains all the essential ideals.

The ring R is Goldie torsion-free if and only if $Z(R) = 0$ by 5.2.9, and in this case we have $T_G \subseteq T_0$ by 5.1.2. Since $T_0 \subseteq T_G$ always by first part of the proof, we have that $T_G = T_0$ if and only if $Z(R) = 0$. □

5.2.13 THEOREM. The Goldie torsion theory coincides with the standard concepts of torsion and torsion-free for modules over an integral domain.

Proof: Let R be an integral domain. We claim $Z(R) = 0$. Let $a \in R$ such that $(0:a) \trianglelefteq R$. Let $0 \neq x \in R$. Then $Rx \cap (0:a) \neq 0$. Hence there exists $r \in R$ such that $rx \neq 0$ and $rxa = 0$. But this implies $a = 0$ and hence $Z(R) = 0$. By 5.2.12 $T_G = T_0$ which we showed coincides with the standard concept of torsion for modules over an integral domain in 5.1.4. □

SECTION 3: Simple Torsion Theory

Let R be a ring and let S be a respresentative set of non-isomorphic simple R-modules; that is, S is a set of simple R-modules no two of which are isomorphic and each simple R-module is isomorphic to some member of S. One should observe that S is closed under homomorphic images and cyclic submodules, and hence

$T_S = LR(S)$ is a hereditary torsion class by 2.7 and 2.8. Also T_S is the smallest torsion class containing S and hence the smallest torsion class containing all the simple modules. The torsion theory associated with the torsion class T_S is called the <u>simple torsion theory</u>, and modules in T_S are said to be <u>simple torsion</u>.

The following theorem will tell us how to compute the filter $F(T_S)$ for the simple torsion class T_S.

<u>5.3.1 THEOREM</u>. Let L be a left ideal of R. Then $L \in F(T_S)$ if and only if whenever L' is a left ideal of R such that $L \subseteq L'$ and $L' \neq R$, then there is $x \in R - L'$ such that $R/(L':x)$ is simple.

<u>Proof</u>: $(\to)$ Let $L \in F(T_S)$ and let L' be a left ideal of R such that $L \subseteq L'$ and $L' \neq R$. Then R/L' is a nonzero homomorphic image of R/L and hence must contain a nonzero simple submodule. Since simple modules are cyclic, the simple submodule of R/L' looks like $Rx + L'/L'$ where $x \in R - L'$. But $Rx + L'/L' \cong R/(L':x)$ and hence $R/(L':x)$ is simple.

$(\leftarrow)$ Assume the condition on L holds. Let M be a nonzero homomorphic image of R/L. Then $M \cong R/L'$ where $L' \leq R$ with $L' \neq R$ and $L \subseteq L'$. There exists $x \in R - L'$ such that $R/(L':x)$ is simple, and $R/(L':x)$ is nonzero since $x \in R - L'$. But $R/(L':x) \cong Rx + L'/L' \subseteq R/L'$, and hence M has a nonzero simple submodule. Thus $R/L \in T_S$ and $L \in F(T_S)$. $\square$

The next theorem gives a necessary and sufficient condition for the simple torsion class T_S to be contained in the E(R)-torsion class T_0.

5.3.2 THEOREM. The simple torsion class T_S is contained in the E(R)-torsion class T_0 if and only if R is T_S -torsion-free; that is, if and only if R has no minimal left ideals.

Proof: ($\rightarrow$) Let $T_S \subseteq T_0$. Then R is T_0-torsion-free and $\mathrm{Hom}(T,R) = 0$ for all $T \in T_0$. Thus $\mathrm{Hom}(T,R) = 0$ for all $T \in T_S$, but this implies R is T_S-torsion-free.

($\leftarrow$) Let R be T_S-torsion-free. Then since T_S is hereditary, by 5.1.2 we have $T_S \subseteq T_0$. □

The next theorem gives us a sufficient condition for the simple torsion class T_S to be contained in the Goldie torsion class T_G.

5.3.3 THEOREM. If R is a ring which has no projective simple modules, then the simple torsion class T_S is contained in the Goldie torsion class T_G.

Proof: Let R be a ring which has no projective simple modules. Let $M \in T_S$ and let N be a nonzero homomorphic image of M. Then N contains a nonzero simple submodule S, and $S \cong R/I$ where I is a maximal left ideal of R. We now wish to verify $I \trianglelefteq R$. Assume there exists $0 \neq H \leq R$ such that

$H \cap I = 0$. Then $R \cong H \oplus I$, and hence $R/I \cong H$ is a summand of a free module and hence projective. Thus we have that H is a projective simple submodule of R and this contradicts the hypothesis. Thus $I \trianglelefteq R$ and $S \cong R/I \in \mathcal{T}_G$. So N contains a nonzero submodule in $\mathcal{G}$, and this implies $M \in \mathcal{T}_G$. Therefore $\mathcal{T}_S \subseteq \mathcal{T}_G$. $\square$

<u>5.3.4 THEOREM</u>. The simple torsion theory coincides with the standard concepts of torsion and torsion-free for modules over an integral domain R when R is the integers.

<u>Proof</u>: Let $R = Z$. Let $M \in {}_R\mathcal{M}$. Let $T(M) = \{x \in M \mid$ there exists $0 \neq r \in R$ such that $rx = 0\}$. Let M_s be the torsion submodule of M in the simple torsion theory. We now wish to verify $T(M) = M_s$. Since $R = Z$ we have that $R \in \mathcal{F}_S$ since the integers contain no minimal left ideals and hence no simple submodules. Thus $\mathcal{T}_S \subseteq \mathcal{T}_0$ by 5.1.2. But $\mathcal{T}_0$ is the standard torsion theory by 5.1.4, so $M_s \subseteq T(M)$. We now want $T(M) \subseteq M_s$. It will suffice to show that $T(M) \in \mathcal{T}_S$; that is, every nonzero homomorphic image of $T(M)$ contains a simple submodule. But since the standard torsion theory is closed under homomorphic images, we only need see that $T(N)$ contains a simple submodule for every $N \in {}_R\mathcal{M}$. Let $x \in T(N)$. Then since $T(N)$ is a torsion abelian group, x has order $n \in Z^+$. But $Z_n \cong Zx \leq T(N)$ and Z_n has a simple submodule. Thus $T(N)$ has

a simple submodule and $T(N) \in \mathcal{T}_S$ for every $N \in {}_R\mathcal{M}$. Hence $T(M) \subseteq M_s$ and we have verified $T(M) = M_s$. $\square$

A more general question unknown to the author is whether or not the simple torsion theory coincides with the standard concept of torsion for modules over any integral domain R whenever R is simple torsion-free. In this case we know $\mathcal{T}_S \subseteq \mathcal{T}_0$ on ${}_R\mathcal{M}$ and thus $M_s \subseteq T(M)$ for all $M \in {}_R\mathcal{M}$. Thus if the simple torsion theory does coincide with the standard concept of torsion, $T(M) \subseteq M_s$ for all $M \in {}_R\mathcal{M}$ which, as in the proof of 5.3.4, means $T(M)$ contains a simple submodule for all $M \in {}_R\mathcal{M}$. We know of no examples for which this is not true.

SUMMARY

In conclusion, we have extended the concept of torsion and torsion-free from abelian group theory to the category ${}_R\mathcal{M}$ of left R-modules over a ring R. To do so we used S. E. Dickson's definition of a torsion theory for certain abelian categories which included ${}_R\mathcal{M}$. We characterized a class $\mathcal{T}$ of modules as a torsion class if and only if $\mathcal{T}$ was closed under homomorphic images, extensions, and arbitrary direct sums. Dually we characterized a class $\mathcal{F}$ of modules as a torsion-free class if and only if $\mathcal{F}$ was closed under submodules, extensions, and arbitrary direct products.

By means of L and R operators we established a way of generating a torsion theory for ${}_R\mathcal{M}$ from an arbitrary class $\mathcal{A}$ of left R-modules, and we proved $LR(\mathcal{A})$ was the smallest torsion class containing $\mathcal{A}$.

We showed there was a one-to-one correspondence between hereditary torsion theories for ${}_R\mathcal{M}$ and torsion filters for R. Thus a natural problem would be to construct all the torsion filters for various rings, but this appears to be very difficult--even for the integers. We found a necessary and sufficient condition for a hereditary torsion class $\mathcal{T}$ to be a TTF class, the condition being that the associated filter $F(\mathcal{T})$ have a smallest element.

We constructed and examined the E(R)-torsion theory, the Goldie torsion theory, and the simple torsion theory. We showed that each construction resulted in a hereditary torsion theory, we found various inclusion relationships between the three, and we computed the filter of each. A remaining problem is to find better characterizations of the filters for the Goldie and simple torsion theories and to perhaps discover other conditions to make the inclusion relationships hold.

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